CHESS AS A COMBINATORIAL GAME

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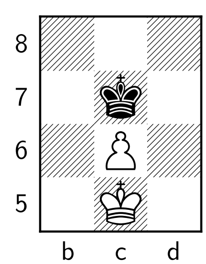

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There are two distinct, though possibly over-lapping, ways of studying chess as a Combinatorial Game; namely, as a mathematician or as a chess player. The first restricts or expands the natural rules or form of the Game, seeking to find simple and beautiful games such as 0, 1, 1/2, $*$, $\uparrow$, etc. This approach has been explored by Noam Elkies in his 1996 paper, *On Numbers and Endgames* [3]. That paper considers primarily positions in which the only mobile pieces are Pawns. To make it a chess game, Kings are placed in such a way that no King move is ever viable. In this way, chess becomes a true Combinatorial Game in the simplest sense. The mathematics is cleaner, but is this analysis useful to a chess player? Since Mutual Zugzwangs are so rare, this type of position is more of a novelty than a practical field of study. Although Elkies demonstrates some practical chess value through his analysis of the endgame in Euwe vs. Hooper [3], his mathematical motivations are clear: his task is to identify known mathematical objects using the rules of chess. Although finding these objects on a traditional 8x8 chessboard is preferable, he drifts from the world of chess in exploring larger boards [3] [4].

In this paper, I take an alternate approach. The goal is to apply Combinatorial Game Theory to general King and Pawn Endings, which are of extreme importance to all serious chess players. We seek to understand the positions which arise naturally on a chessboard as combinatorial games. This will require tweaking the mathematical interpretation of the scenario, rather than restricting the chess positions. This paper is only a modest beginning. In particular, we consider here positions with 2 Kings alone, and 2 Kings and a single Pawn, played on a board no larger than 3x4. This work could easily be expanded to a larger board with a little time, and could quite likely be expanded to include more pieces with computer assistance. It is not clear whether this approach actually simplifies the analysis of a given position, but it is certainly a new vantage point from which to approach the question.

It is well-known that Chess is not a priori a Combinatorial Game since, in Elkies' words, "the winner of a chess game is in general not determined by who makes the last move, and indeed a game may be neither won nor lost at all but drawn by infinite play" [3]. Elkies avoids this problem by crafting situations where the winner *is* determined by who makes the last move and avoiding any chance of infinite play. In our approach, we must be creative in order to avoid these barriers in the study of natural chess positions.

In Section 1, we introduce the basic notion of Combinatorial Game Theory. In Section 2, we study a class of simple games called Enders, although we are generally interested in more complex games. These first two sections can be seen as a new presentation of the material in Winning Ways, Volume 1. Section 3 contains no mathematics, but rather exposes the reader to the basic chess concepts relevant to the positions we later analyze. In Section 4, we generalize Section 2 from Enders to loopy games. This section uses definitions from Winning Ways, Volume 2. Finally, in Section 5, we turn to our object of interest: chess endgames.

1. Introduction: What is Combinatorial Game Theory?

Combinatorial Game Theory analyzes turn-based games with two players (often called Left and Right). A general **game**, G can be written in the form

$$G = \{G^L | G^R\}.$$

Here G^L is a collection of games which are Left's **options**. That is to say, if it were Left to Move (LTM), Left would move from G to any one of G^L. G^L may refer to a single Left option, the empty set, or even an infinite set.

We will adopt the **Normal Play Convention**, under which a player loses if there are no legal moves.[1] This is the only way play ends. We assume that games satisfy the following **ending condition**: There does not exist an infinite **play** of alternating moves

$$G, G^L, G^{LR}, G^{LRL}, \ldots$$

It follows that there are no ties.[2] We also assume that both players have complete information and that there are no random variables involved. These latter assumptions allow for the determination of a **strategy** which will ensure a favorable **outcome**.

We now introduce Hackenbush, a **Game** we will consider more in Sections 2.3 and 2.4. A game in this Game consists of a collection of edges which are either b**L**ue or **R**ed. Each edge is connected to the ground, either directly or indirectly. On Left's turn, Left removes one bLue edge. Any edges which are left disconnected from the ground disappear from play. In response, Right removes one Red edge, and any disconnected edges disappear. As usual, the loser is the person who is unable to move.

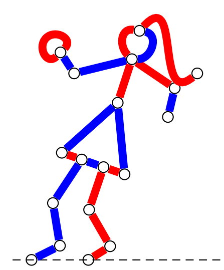


FIGURE 1. Would you rather play as Left or Right? [1, p. 1]

[1]One could also consider the natural opposite of this convention, called the Misère Play Convention. A player loses by making the final move in a Misère game.

[2]The first barriers of applying CGT to chess appear. Chess does have ties, does not observe the Normal Play Convention, and preventing infinite plays requires complex rules such as *Draw by Repetition* and *The 50 Move Rule*.

Observe that we use **Game** differently than the more familiar notion of a **game** explained above. We will use the upper-case **Game** to mean the formal rules which specify which options are allowable from each position. Examples of Games include Hackenbush, Chess, and Tic Tac Toe. We will use the lower-case **game** to refer to a single position. For example, we might like to understand the game in Figure 1.

1.1. **Outcome Classes and Addition.** As mentioned earlier, our assumptions that

(1) games follow the Normal Play Convention,
(2) games have no random variables,
(3) games terminate after finite time, and
(4) players have complete information

allow for the determination of the **Outcome Class** of a game, assuming best play. In particular, given a game together with an assignment of who should move first, exactly one player has a winning strategy. Since there are only two possibilities of who can move first and two possibilities of who can win, there are $2 * 2 = 4$ Outcome Classes, defined as follows [1, p. 29].

Definition 1. *A game is* **positive** *if Left has a winning strategy, regardless of who moves first.*

Definition 2. *A game is* **negative** *if Right has a winning strategy, regardless of who moves first.*

These first two definitions describe a game in which one player has a clear advantage. The next two definitions describe games in which neither player is clearly winning. That is, the outcome depends on who moves first.

Definition 3. *A game is a* **zero game** *if the second player has a winning strategy.*

Definition 4. *A game is* **fuzzy** *if the first player has a winning strategy.*

All four of these definitions will come to have a more numerical interpretation, but for now they are purely game theoretic.

Theorem 5. *Let G be a game which satisfies the assumptions listed at the beginning of this section. Then, G belongs to exactly one outcome class.* [1, p. 46]

Proof. Certainly G does not belong to more than one outcome class, as it is impossible for two opposing players to simultaneously have a winning strategy.

Suppose G does not belong to any outcome class. Without loss of generality, assume that neither player has a winning strategy when Left is to move. Then, since Left can not force a win, there are no options G^L which are forced wins. But since Right can't force a win, there exists G^L such that Right can not force a win. Then, we have shown that the absence of a winning strategy implies that there exists an option to move to a game without a winning strategy. It is obvious that such a game can last forever, which contradicts the ending condition. □

It is worth observing that a game in Combinatorial Game Theory is analyzed by considering both players' options—not just the options of the player at move. This consideration allows for independent games to be combined in a natural way:

$$G + H = \{G^L + H, G + H^L | G^R + H, G + H^R\}$$

A move in the sum $G+H$ is *either* a move in G *or* a move in H. It is easy to see that this addition is associative and commutative. Some care is required, because it's not clear that $G + H$ satisfies our assumptions. $G + H$ is still a 2 player, turned based game which obeys the Normal Play Convention, but it does not necessarily satisfy the ending condition.

Example [1, p. 47] Let G and H be games. Suppose that from G there exists a string of Left options

$$G, G^L, G^{LL}, G^{LLL}, ...$$

and from H there exists a string of Right options

$$H, H^R, H^{RR}, ...$$

Then, even if G and H satisfy the ending condition, $G + H$ does not, since the following play does not terminate:

$$G + H, G^L + H, G^L + H^R, G^{LL} + H^R, G^{LL} + H^{RR}, ...$$

There are two ways to study the additive structure of an object which is not closed under addition. The first is to consider its additive closure, and the second is to consider a subset which is closed under addition. We will introduce the first approach in Section 4. There, we will call G and H **Stoppers** and $G + H$ a **loopy game**. However, in order to understand these loopy games, we must first understand games that add naturally; that is, games where $G + H$ satisfies our ending condition. To ensure this, we just need to impose a stricter ending condition. Our earlier ending condition required that there is no infinite, alternating play $G, G^L, G^{LR}, G^{LRL}, ...$ We will refer to this as the **Weak Ending Condition**. In Section 2, we instead require that there is no infinite play, *alternating or not*. This latter condition will be called the **Strong Ending Condition** and games which satisfy it are called **Enders**.

2. **The Structure of the set of Enders**

2.1. Properties of Addition.

Definition 6. *A game is an Ender if it satisfies the Strong Ending Condition.* [1, p. 329]

We will use $\mathbb{E}$ to denote the set of Enders.

Proposition 7. $\mathbb{E}$ *is closed under addition.*

Proof. Suppose G and H satisfy the Strong Ending Condition, but $G + H$ does not. Then, there exists an infinite play in $G + H$. Since each move in the sum was either in G or H, either the total play in G or the total play in H must have been infinite. But this is a contradiction since G and H are Enders. □

Given the outcome class of G and H, can we identify the outcome class of $G + H$? The following propositions answer this question [1, p. 32].

Proposition 8. *If $G, H \in \mathbb{E}$ are both zero games, then $G + H$ is a zero game.*

Proof. The second player simply responds in whichever component the first player moved. It is obvious that this strategy is non-losing. Since $G + H$ cannot last forever by assumption, $G + H$ is a zero game by Theorem 5. □

The next proposition generalizes this conclusion about zero games:

Proposition 9. *Let $G \in \mathbb{E}$ be a zero game, and let $H \in \mathbb{E}$ be any game. A winning strategy in H extends to a winning strategy in $G + H$.*

Proof. Without loss of generality, assume Left has a winning strategy in H when Right is at move. Consider $G + H$ when Right is at move. If Right moves in $G \mapsto G^R$, Left wins with a response in the same component. Such a response exists because G is a zero game. If Right moves $H \mapsto H^R$, Left's winning response is $H^R \mapsto H^{RL}$. Such a response exists because Left has a winning strategy in H when Right is at move. We have shown that Left has a response to any given Right move and thus, a non-losing strategy. Since $G + H$ cannot last forever by assumption, this strategy is winning. □

The next proposition can be similarly proven for negative games.

Proposition 10. *If $G, H \in \mathbb{E}$ are both positive, then $G + H$ is positive.*

Proof. Left has a winning strategy in either component, and so plays them completely separately. Left starts with a winning move in either component and then uses the obvious strategy. □

What else can be said about addition in $\mathbb{E}$? As we might hope, $\mathbb{E}$ is a group. We have associativity, so we need the existence of 0 and negatives. First, we need to define the latter two concepts. Since we have not yet defined what it means for two games to be equal, the following definition relies on Outcome Classes.

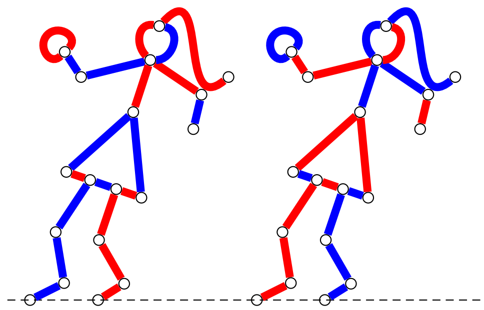


FIGURE 2. A simple strategy for the second player

Definition 11. *Suppose G and H are games and $G+H$ is a zero game. Then, H is the* **negative** *of G, written $(-G)$.*

How can we find a negative of a game? Given a game G, construct G' by interchanging the role of Left and Right. G' satisfies the Strong Ending Condition, since an infinite play must have also been infinite in G.

Proposition 12. *Let $G \in \mathbb{E}$ and construct G' by reversing the role of each player (see Figure 2). Then, $G+G'$ is a zero game. In other words, G' is a negative of G.* [1, p. 34-35]

Proof. We must check that the second player has a winning strategy. The second player can always find a playable move, simply by mirroring the first player in the alternate component. Then, $G+G'$ is a zero game by Theorem 5. □

The following table summarizes the outcome class of $(-G)$.

TABLE 1. The outcome of $(-G)$

G	$(-G)$
positive	negative
negative	positive
fuzzy	fuzzy
zero game	zero game

2.1.1. *Playing games with kids.* The strategy for the second player of mirroring the first player's moves, as in the proof of Proposition 12 comes up again and again. Winning Ways refers to it as the Tweedledum and Tweedledee strategy [1, p. 3]. As a scholastic chess teacher, a similar strategy comes up in some Pawn Games that I use. With my students, I call it "Copycat," and I will use that language throughout this paper instead of Tweedledum and

Tweedledee. If you don't like playing games with kids, you can skip the rest of this section. If you do, consider the position in Figure 3.

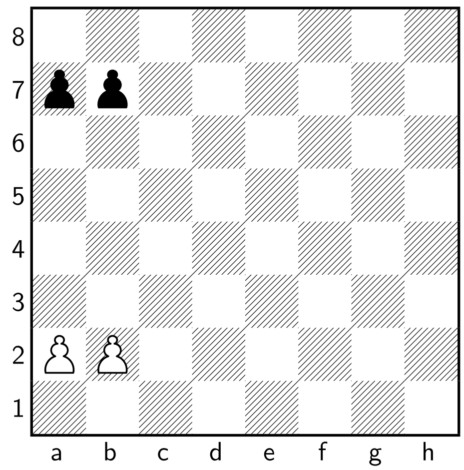


FIGURE 3. 2 v. 2 Pawn Game

As my students trickle in for after school chess club, they find this position set up on a board. They play it amongst themselves, once or twice with each color, to get a sense of how it works. Most just start moving pieces, assuming they win by taking their opponent's Pawns, but a few are goal-conscious enough to ask, "How do you win?"

"By getting a Pawn to the other side first or by taking all of your opponent's Pawns."

After everyone has arrived, gotten settled, and had a chance to play, I have them leave the boards they've been playing at and take a seat in front of a Demo Board (a large, hanging chess board for use by a group) to discuss the game they've been playing. When I bring them back together, I say, "Raise your hand if you won with White." A few hands go up.

"Raise your hand if you got a draw." About an equal number of hands go up.

"And raise your hand if you won with Black." A much higher number of hands go up.

"Whoa! A lot of you won as Black! Do you think Black can always win this game?"

"No! It depends," a student emphatically responds. After asking them to clarify what they mean by "it depends," I almost always hear that it depends on what moves Black makes. And at this point, I introduce a challenging and foreign concept to these young children: there is such a thing as "best play" in a game.

I ask them, "If both sides make the best possible moves, how will this game end?" With the option of "it depends" now off the table, the children's

1 b4 b5

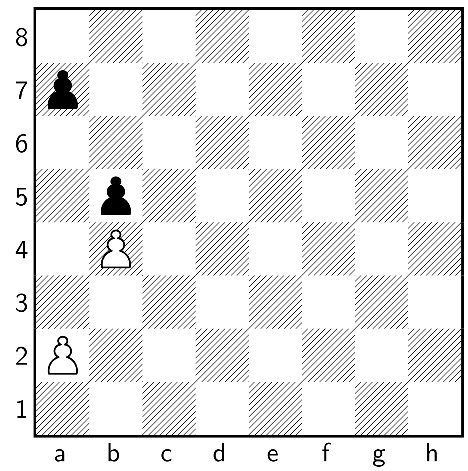


FIGURE 4. Black is copying

answers are spread between the three remaining possible answers. I call on someone who correctly and confidently said that Black would win.

"You're right! Black can force a win. I'll let you play Black...think you can beat me?" I ask the student. Using the Demo Board, I play as White with the student playing as Black. Usually, I manage to draw or win.

"Wait a minute—didn't I just say Black would win? What gives?"

"I didn't make the right moves." The students come to realize that even though I told them Black has an advantage, White will still be able to win if Black doesn't react correctly. I introduce the notion of a **strategy**, explaining that if they learn this technique, they can win as Black every time. Students are exposed to the idea of being careful and planning ahead, rather than playing on impulse and instinct. The strategy, Copycat, is pretty straightforward. Whatever White does with the a-pawn, Black will do with the a-pawn. The same goes for the other pair of pawns. In this example, White begins by pushing the b-pawn two squares, so Black pushes the b-pawn two squares as well (See Figure 4).

Now White can not move the b-pawn and **2.a4** hangs the pawn, so White pushes the a-pawn up one square. Sticking with the Copycat strategy, Black does the same. And now White will lose the Pawns, one followed by the other (although Black could also win without taking the second Pawn, simply by winning the race to the other side).

And finally, the students get a chance to put this strategy into action. If they use the Copycat strategy as Black, they will reach Figure 5 by force. But if they are not being careful and just blindly copying, after **a3–a4**, Black will respond with **. . . a6–a5**, costing them the game. I let them discover this caveat to the Copycat strategy on their own in order to show them that it

2 a3 a6

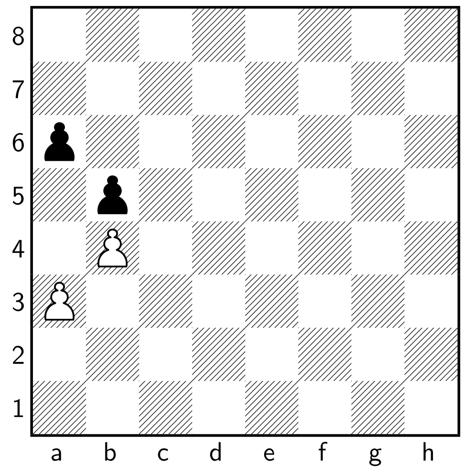


Figure 5. White will lose the Pawns

is important to always be thinking and questioning, rather than passively following authority.

The strategy is not exactly the same as Tweedledum and Tweedledee. In particular, this Pawn Game does not appear to be the sum of two simpler components. Conceptually, however, the two strategies are very similar. In both cases, the second player has a simple and direct plan to victory by mirroring his opponent's moves. Eventually, this will lead to a zero game, as in Figure 5. We now re-join our pedophobic readers.

2.1.2. *Playing games without children.* The citations in the remainder of this section refer to the relevant pages in Winning Ways, but the presentation differs slightly. In particular, the presentation here uses the definition of $-G$ as the most elementary building block—the point from which everything begins. On the other hand, by the time Winning Ways introduces the negative [1, p. 33], they have already gotten their hands dirty, finding a copy of $\mathbb{Z} \subset \mathbb{E}$ as well as the dyadic fractions [1, p. 19-20]. I will refrain from giving such examples until the group structure of $\mathbb{E}$ has been made precise, in order to avoid any abuse of the terms 0 and $=$.

Having described the notion of a negative of a game, we are now prepared to define equality:

Definition 13. *Let $G, H \in \mathbb{E}$. Then, $G = H$ if $G + (-H)$ is a zero game.*

Observe that $(-H)$ exists by the role-reversing construction above.

Theorem 14. *$=$ is an equivalence relation on $\mathbb{E}$.* [1, p. 35]

Proof. $G + (-G)$ is a zero game by definition, so $=$ is reflexive.

Suppose $G = H$. To show $=$ is symmetric, we need $H = G$. Equivalently, we need to show $H + (-G)$ is a zero game given that $G + (-H)$ is a zero game. This is the same as checking that a negative of a zero game is a zero game, which is obvious.

Suppose $A = B$ and $B = C$. We need to show $A + (-C)$ is a zero game. By Proposition 9, we can add a zero game without affecting the equivalence class. Then,

$$A + (-C) = [A + (-B)] + [B + (-C)] = 0 + 0 = 0,$$

so $=$ is transitive. □

Proposition 15. $A = B \implies$ *A and B are in the same outcome class. In other words, this equivalence relation respects outcome classes.*

Proof. Suppose A and B are in different outcome classes. Assume without loss of generality that the outcome differs when Left starts. For sake of argument, suppose that when Left starts in both games, Left can win A but not B. We want to see that $A \neq B$. Equivalently, $A + (-B)$ is not a zero game. If Left starts, Left can win by simply playing a winning move in A. Now, Right has to move in one component or the other, but he is unable to find a move in either one which will prove advantageous. Left will simply respond in whichever component Right plays in. Since this strategy would win each component individually and the components are independent, this strategy wins for Left. □

We now present the usual notion of the additive identity from elementary Algebra.

Definition 16. *A game* $e \in \mathbb{E}$ *is called an additive identity if for each* $G \in \mathbb{E}$, $G + e = G$.

The following theorem validates the term "zero game."

Theorem 17. $G \in \mathbb{E}$ *is a zero game* $\iff$ G *is an additive identity.* [1, p. 32]

Proof. Suppose $G \in \mathbb{E}$ is a zero game, and $H \in \mathbb{E}$ is any game. To show the forward direction, we need to see that $G + H = H$ or that $G + H + (-H)$ is a zero game. This follows from Proposition 8 since G and $H + (-H)$ are both zero games.

Suppose $G \in \mathbb{E}$ is an additive identity, and $H \in \mathbb{E}$ is a zero game. Using the forward direction and Table 1, we have that $G = G + (-H) = H$. Then, Proposition 15 implies that G is a zero game. □

It is easy to see that all zero games are equal, since we know from Algebra that the additive identity is unique. This also follows immediately from Theorem 17, Proposition 8 and Table 1. When G is a zero game, we will write $G = 0$.

We have now shown that $\mathbb{E}/=$ (the set of Enders, modulo the equivalence relation $=$) is an abelian group. It would be natural to explore this structure further: what are the subgroups and quotients of $\mathbb{E}$? We will not consider such questions. For chess analysis, we are not so interested in the additive structure. Indeed the conception of adding two chess positions makes little sense if Kings are involved. Rather, in chess we are more interested in evaluating decisions and comparing options.

2.2. **Comparing Enders.** [1, p. 35]

Definition 18. *Let $G, H \in \mathbb{E}$. We say G is* **less than** *H if $G + (-H)$ is negative. We write $G < H$.*

One could similarly define what it means for G to be **greater than** H (written, of course, $G > H$). Since the negative of a negative game is positive, $G < H \iff H > G$.

Definition 19. *Let $G, H \in \mathbb{E}$. G is* **confused with** *H if $G + (-H)$ is fuzzy. We write $G||H$.*

Theorem 17 gives a natural correspondence between these definitions and our earlier notions of **zero game, positive, negative, and fuzzy**:

(1) $G = 0 \iff G$ is a zero game.
(2) $G > 0 \iff G$ is positive.
(3) $G < 0 \iff G$ is negative.
(4) $G||0 \iff G$ is fuzzy.

Corollary 20 (Quadrichotomy)**.** *Let $G, H \in \mathbb{E}$. Then, exactly one of the following is true:*

(1) $G = H$,
(2) $G < H$,
(3) $G > H$, *or*
(4) $G||H$

Proof. This is a Corollary to Theorem 5, which says that $G + (-H)$ belongs to exactly one Outcome Class. □

We write $G \lhd| H$ to mean that G is less than or fuzzy with H and $G \leq H$ to mean that G is less than or equal to H. . The following theorem validates the definition of $\leq$.

Theorem 21. *$\leq$ is a partial order on $\mathbb{E}$.*

Proof. Reflexivity of $\leq$ follows immediately from that of $=$.

To check anti-symmetry, suppose $G \leq H$ and $H \leq G$. Then, $G = H$ by Corollary 20.

To check transitivity, suppose $A \leq B$ and $B \leq C$. That is, $A + (-B) \leq 0$ and $B + (-C) \leq 0$. Adding $(-B) + B$ is just adding 0, so we can write

$$A + (-C) = A + [(-B) + B] + (-C).$$

Addition is associative, and after regrouping we recognize that

$$A + (-C) = [A + (-B)] + [B + (-C)]$$

is the sum of two games which are either positive or zero. Then, $A+(-C) \leq 0$ by Propositions 9 and 10. Therefore, $A \leq C$ as desired. □

Observe that $\leq$ is not a total order—that is, there exist $G, H \in \mathbb{E}$ such that $G \not\leq H$ and $H \not\leq G$. In particular, we cannot use $\leq$ to compare games which are **confused** with each other.

With well-defined notions of adding and comparing Enders, it is now time to see a few.

2.3. **The Integers as games.** Recall the rules of Hackenbush from page 6.

Figure 6. A game that has ended

Consider first the position with no edges, shown in Figure 6. There are no edges, so neither player is able to move. Then, this is a zero game, since the first player loses (recall that we use the Normal Play Convention). Therefore, we write

$$G = \{G^L | G^R\} = \{\ \ |\ \ \} = 0.$$

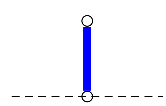


Figure 7. Left has an edge

In Figure 7, there is one Blue edge and no Red edges. Formally, we can write this as $\{0|\ \ \}$. This is the first positive game we have seen, so let's call it 1. Similarly, any position with n non-stacked Blue edges is $\{n-1|\ \ \} = n$ and positions with n non-stacked Red edges as $\{\ \ |1-n\} = -n$ [1, p. 19].

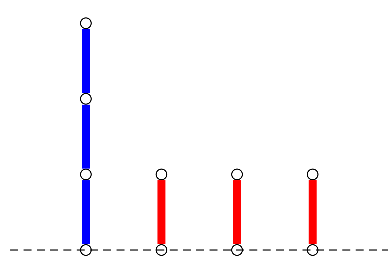


Figure 8. A zero game

Observe that Figure 8 is a zero game. It is not so surprising that a string of three is the same as three separate edges, but it is not entirely obvious that Left's extra options to simultaneously eliminate multiple edges does not affect the equivalence class. We formalize this notion in Proposition 26.

Addition on Hackenbush agrees with addition on the integers. We can write $n + (-n) = 0$ since a position with an equal number of (independent) edges is a zero game. Similarly, we can say that $5 + (-3) = 2$ since the Hackenbush position represented by $5 + (-3) + (-2)$ is a zero game. We will consider a wider variety of Hackenbush positions in Section 2.4.

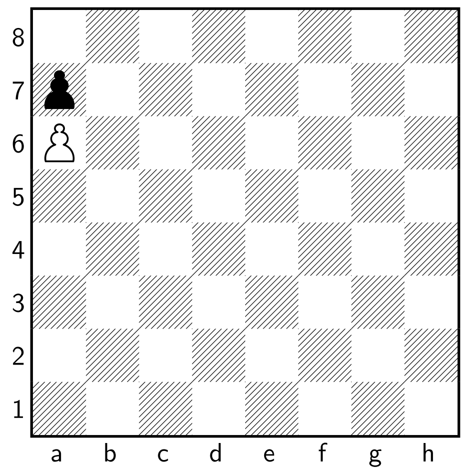


FIGURE 9. 0 on a Chessboard

We can construct the integers on a chessboard just as easily as in Hackenbush. Since neither player can move, the game shown in Figure 9 is 0. Adding a White pawn at a3 gives Figure 10 which we recognize as $\{1|\ \} = 2$ [3, p. 3].

But this can hardly be described as chess; there are no decisions to be made, and there aren't even Kings on the board. Adding Kings could devastate the simplicity of these positions, but if we can create a zero game with Kings, Theorem 17 says that we can add it with no effect.

In the chess literature, a zero game is already a well-known concept, although it goes by a different name. This phenomenon, **Mutual Zugzwang** or MZZ, is rare in chess, but it is common enough to be in every tournament player's vernacular.[3]

The most famous example of MZZ is the trébuchet, shown in Figure 11. Whoever is at move must move the King. But moving the King will leave the Pawn undefended. Then, whoever moves loses since the second player will win the Pawn and then promote his own Pawn. This game is therefore a zero game. Because these King moves are not viable at all, we will disallow them and write the trébuchet as $\{\ |\ \}$.

[3]Less rare is zugzwang, in which only one player is unable to find a viable move.

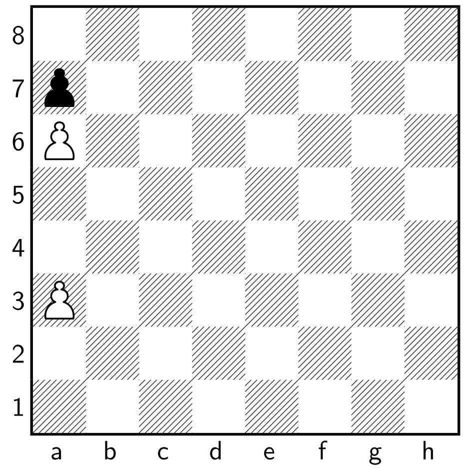


FIGURE 10. 2 on a Chessboard

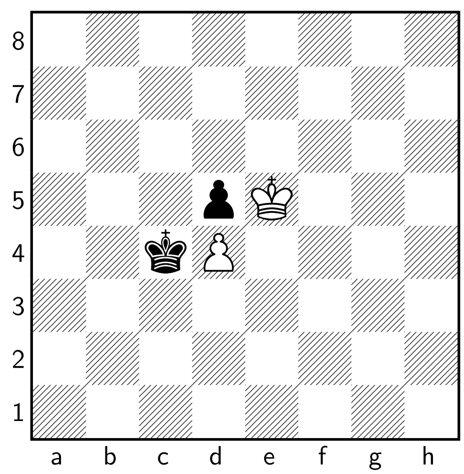


FIGURE 11. A trébuchet

This is the basic recipe for the current applications of Combinatorial Game Theory to Chess. The Kings are placed in MZZ (usually a trébuchet), while some independent pawn formation is unclear. The player who is unable to move in the Pawn formation will be forced to move the King, which amounts to losing the game. Elkies explores this interesting class of positions in *On Numbers and Endgames*.

To a chess player, however, this class of positions is more of a novelty than a practical field of study. What applications does CGT have to more essential chess positions? Endgames will certainly be the most realistic place to focus, but perhaps we can allow pieces other than Pawns to make moves. Can we describe King maneuvering concepts such as *Opposition* and *Triangulation* using CGT? Can CGT shed light on the winning chances in positions with a King and a Pawn against a King? Before considering these questions, let's continue our exploration of the class of games, drawing examples from Hackenbush and Chess.

2.4. **The Surreal Numbers.** Conway gives the following formal construction of the Surreal Numbers in the first pages of *On Numbers and Games.*

Construction 22. *If G^L, G^R are sets of Surreal Numbers and no member of G^L is greater than or equal to any member of G^R, then there is a Surreal Number $\{G^L|G^R\}$. All Surreal Numbers are constructed in this way.* [2, p. 4]

Observe that the additive structure of the Surreal Numbers is built into the construction because without it we could not understand $\geq$. Conway defines the partial order slightly differently, offering a more numerical and less game theoretic approach than the presentation here. Regardless, both definitions of $\geq$ give rise to the same Surreal Numbers. Conway gives further structure by defining multiplication, allowing us to view the Surreal Numbers as a field. Here, we will not consider the multiplicative structure.

The Surreal Numbers form a totally ordered extension of $\mathbb{R}$ and a subgroup of $\mathbb{E}$. Since $\mathbb{E}$ is commutative, we could consider taking quotients of $\mathbb{E}$. This is another topic which we will not consider.

In this section, we consider a more hands-on introduction to the Surreal Numbers, by considering Hackenbush positions. We showed in Section 2.3 that there is a correspondence between some Hackenbush positions and the integers. More precisely, let Z be a set of representative elements for each equivalence class of Hackenbush positions in which all edges touch the ground. Then, $Z \cong \mathbb{Z}$. Do isomorphic copies of $\mathbb{Q}$ or $\mathbb{R}$ appear? In this section, we will find a copy of the dyadic fractions inside $\mathbb{E}$. In Section 2.5, we will find a copy of $\mathbb{R}$ and a copy of the Ordinal Numbers in $\mathbb{E}$.

Example

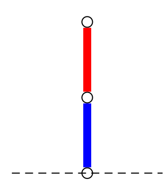


FIGURE 12. Our first fraction

Consider the game in Figure 12, which we will call G. Since each player only has one legal move, we can certainly describe G as shown in Figure 13. Since we already understand G^L and G^R in Figure 13, we can write $G = \{0|1\}$. G is positive since Blue (Left) is winning. But $G < 1$, since $G + (-1)$ (Figure 14) is negative.

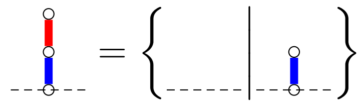


FIGURE 13. Breaking down Figure 12

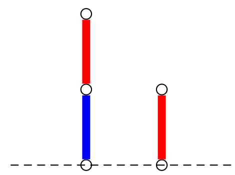


FIGURE 14. $G + (-1) < 0$

After a lucky guess, we can verify that $G = \frac{1}{2}$ by demonstrating that Figure 15 ($G + G - 1$) is a zero game [1, p. 4].

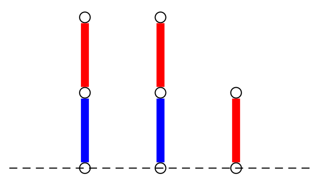


FIGURE 15. $G + G + (-1) = 0$

To see this, observe that in Figure 15, Red has 2 edges which are "en prix"[4], and he will only be able to save one of them. After a reasonable move from each player, we reach Figure 16 which we recognize as 0.

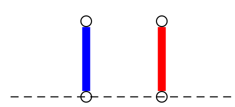


FIGURE 16. Figure 15 after a couple of moves

Example Adding another Red edge to the top of G gives us the game shown in Figure 17, which we will call H. Intuitively, H should be less than $\frac{1}{2}$ since we just added a Red edge to G. We check this by observing that Right can win regardless of who starts in Figure 18. If Right starts, he wins by removing one edge from the first string, leaving a win by Copycat. If Left starts it is even easier.

[4]That is to say, 2 edges which are at risk of disappearing.

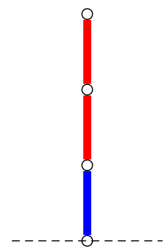


FIGURE 17. What is H?

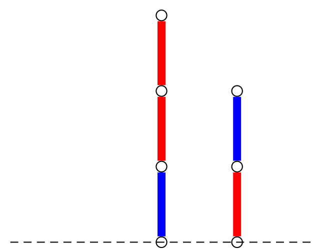


FIGURE 18. $H < 1/2$

But H is certainly positive; Red does not even have an edge touching the ground, so he certainly will not make the final move. After bounding H between 0 and $\frac{1}{2}$, another inspired computation confirms that $H + H + (-\frac{1}{2}) = 0$, so $H = \frac{1}{4}$ [1, p. 6].

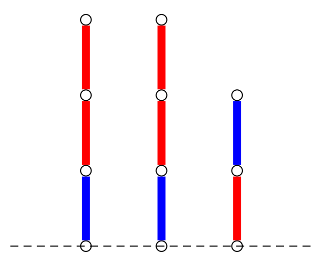


FIGURE 19. $H + H + (-\frac{1}{2}) = 0$

To see this, we need to see that all first moves in Figure 19 are losing. Figure 20 shows the Left and Right options from Figure 19. It is not hard to see that Left's options are negative while Right's options are positive.

We checked above that (a) is negative. (b) is also negative, since $H+H-1 < \frac{1}{2}+\frac{1}{2}-1 = 0$. In (c), we can see the outer strings form a zero game and the middle one is positive, so (c) is positive. In (d) we have $1+H-\frac{1}{2} = H+\frac{1}{2}$ which is positive since both of the summands are positive. And lastly, (e) is positive because only Blue edges touch the ground. This shows that $H = \frac{1}{4}$.

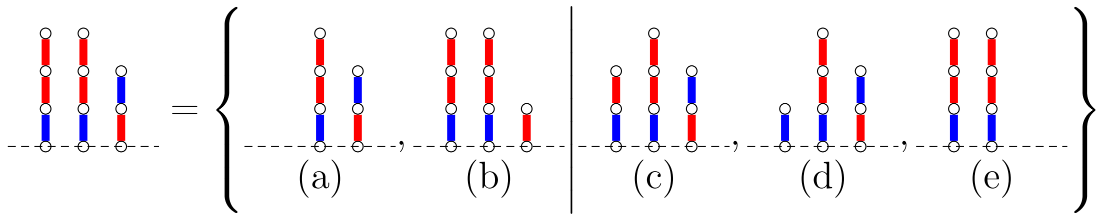


FIGURE 20

We can similarly find a game for any $\frac{1}{2^n}, \forall n \in \mathbb{N}$ by placing n red edges on top of a single blue edge. Combining games of this form gives us all dyadic fractions [1, p. 20]. Such dyadic numbers can also be found on a chessboard, but it requires more ingenuity. Elkies shows examples of $\frac{1}{2}$ and $\frac{1}{4}$ on a normal chessboard and demonstrates the existence of arbitrarily small dyadic fractions on longer boards [3].

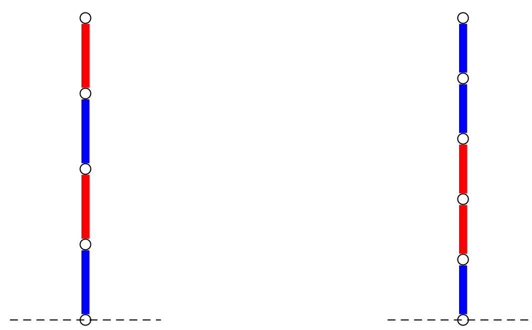


FIGURE 21. More colorful dyadic Hackenbush positions

Without thinking outside the box a little bit, it seems this is all we can get from Hackenbush. Nothing is gained by considering strings such as those in Figure 21; these strings will still have dyadic values. However, if we allow strings of infinitely many edges, we discover much more.

2.5. $\mathbb{R}$ **and On.** In this section, we consider Hackenbush positions with infinitely tall strings. Rather than the previous section's diagram-based approach, we will revert to the notation $G = \{G^L|G^R\}$ where now G^L and/or G^R may be infinite sets of numbers.

The integers can be written

$$n = \{n-1| \quad \}.$$

But in fact, we could change G^L from a singleton to a finite set, without any problems. In fact, the following presentation shows the string of n blue edges, which we know to be n :

$$n = \{0, 1, ..., n-1| \quad \}.$$

Allowing G^L and/or G^R to be infinite sets (or equivalently, allowing infinitely tall Hackenbush strings) is a single construction which subsumes both Dedekind's construction of $\mathbb{R}$ and Cantor's construction of the Ordinal Numbers, **On**. Given the notation, it is not so surprising that Dedekind cuts fit well. Dedekind's assumption that $G^L < G^R$ is built into the definition of a Surreal Number. Then, we see that all of the cuts which Dedekind made in $\mathbb{Q}$ can be viewed just as easily in $\mathbb{E}$ or any other set which is dense in $\mathbb{R}$. And yet, this theory is more general than Dedekind's because we allow for G^R to be empty. In fact, when G^R is empty, we find the Ordinal Numbers [2, p. 3-4].

Observe that these games satisfy the Strong Ending Condition: each player may choose between an infinite number of games, but since each of them is in $\mathbb{E}$, the game itself is in $\mathbb{E}$. In an infinite Hackenbush string, it is easy to see that any play will end in a finite but unbounded number of turns, since the string will be finite after the first turn. It would be problematic, however, to have infinitely many strings.

Consider $G = \{0, 1, ...| \ \ \} = \{\mathbb{Z}^+| \ \ \}$, an infinite string of Blue edges. G is a Surreal Number since there is not even an element of G^R to violate the inequality condition. How does G compare to the numbers we know? If x is any finite, positive number, then $G + (-x)$ is positive, since Left will simply move to $G^L > x$. In fact, $G = \omega$, where ω is the first Ordinal Number. We also have $\omega + 1 = \{\omega| \ \ \}$, $2\omega = \omega + \omega = \{\omega, \omega + 1, ...| \ \ \}$. We can continue defining Ordinal Numbers $(\omega^2, \omega^\omega, \omega^{\omega^\omega}, ...)$ indefinitely, and they will all be Enders [1, p. 329].

Abstractly, we have already found $\mathbb{R}$ by Dedekind's method. It is not so hard to get our hands on these numbers as Hackenbush positions. Berlekamp's Rule gives an explicit construction for any given real number, by creating a sequence of finite Hackenbush strings which converges[5] to it (Berlekamp's method is based on the binary expansion of $x \in \mathbb{R}$) [1, p. 77].

Between $\mathbb{R}$ and **On**, we have constructed most of the Surreal Numbers.[6]

2.6. Working with Numbers.

We saw in Section 2.4 that $\{0|1\} = \frac{1}{2}$ and $\{0|\frac{1}{2}\} = \frac{1}{4}$. What about $G = \{1.25|2\}$? We might guess $G = 1.625$ by taking the average. But in fact, $G < 1.625$, since the sum $G - 1.625{=}G + \{-1.75|-1.5\}$ is negative. Right can win by moving to $-1.5 = \{-2|-1\}$. Then, Left will have to choose between moving to $1.25 + (-1.5)$ or $G + (-2)$, both of which are losing. In fact, $G = 1.5$, as we shall see shortly.

[5] The notion of a sequence of Hackenbush strings converging is somewhat imprecise, since we do not have a metric on $\mathbb{E}$. For this particular example, there is no problem since we can simply define $d_{Hackenbush}(x, y) = d_{\mathbb{R}}(x, y)$ since we have an isometry between finite Hackenbush strings and real numbers. We will not pursue this subject further here, but it is possible to generalize Analytic concepts to infinite strings.

[6] The reciprocals of the Ordinal Numbers are also Surreal Numbers, but they are not of interest to us.

The following theorem describes how to evaluate games such as $G = \{1.25|2\}$ when $G = \{G^L|G^R\}$ is a Surreal Number. Note that we don't actually assume G is a Surreal Number. Rather, if G is a game with some property, we show that G is a Surreal Number and determine which Surreal Number it is.

Theorem 23 (Simplicity Theorem). *Assume $G = \{G^L|G^R\}$ and that for some Surreal Number z, $G^L \not\geq z \not\geq G^R$. Assume further that no option*[7] *of z satisfies this condition. Then, $G = z$.* [2, p. 23]

Proof. Play $G+(-z)$. Left's move to G^L is losing because we have $G^L+(-z)$ is not positive or zero. Similarly, Right's move to G^R is losing because $G^R + (-z)$ is not negative or zero. Without loss of generality, any Left option from z cannot satisfy $G^L \not\geq z^L \not\geq G^R$. Since $z^L < z$, we must have $G^L \geq z^L$. Then, Right's move in the $(-z)$ component to $-z^L$ is losing since Left would respond by moving the total game to $G^L + (-z^L)$ which is positive or zero. □

For the integers, this notion of simplicity amounts to the distance from 0. For the dyadics, a number with a smaller denominator is simpler. In the example above, $G = \{1.25|2\}$, we recognize that for $z = 1.5 = \{1|2\}, 1.25 \not\geq z \not\geq 2$, but that the options z^L and z^R (1 and 2) do not satisfy this condition. Therefore, $G = 1.5$. On the other hand, $G \neq 1.625$ since the Left option 1.5 satisfies this condition [1, p. 21].

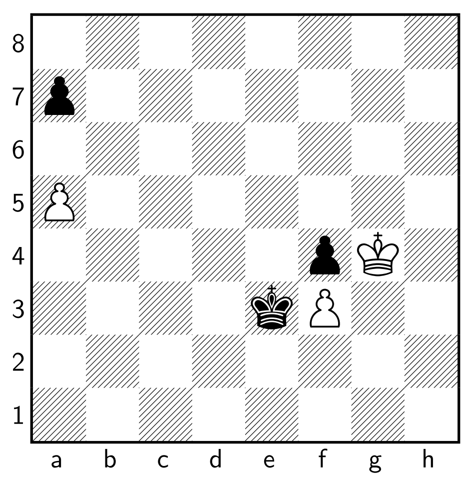


FIGURE 22

2.7. **Moving Away from Numbers.** As we saw in Construction 22, a Surreal Number (or just a number) is a game where $G^L < G^R$. We have seen plenty of examples of numbers, including $\{\ |\ \} = 0, \{0|1\} = \frac{1}{2}$, and

[7]Recall that any element of G^L is called a Left *option*.

$\{\mathbb{Z}|\ \} = \omega$. What new games do we find if we drop this assumption? The following examples introduce some important games, as well as demonstrate the possibility of viewing the same game in several different forms.

Example Let G be the game shown in Figure 22 [3]. Recall that the Kings are disallowed from moving since they are in a trébuchet. Then, each player has only one viable move, so we can write $G = \{0|0\}$. If you're not a fan of chess, you could also think of G as a position in Hackenbush with one edge, but either player may take it. We will use green edges in this manner, and they will be important in Section 2.9 when we discuss Nim. Since moving to 0 guarantees a win, G is fuzzy. We have not previously found any fuzzy games, so we will call this game $*$. It is easy to check that $* + * = 0$ by observing that Figure 23 is a zero game.

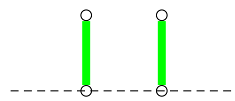


FIGURE 23. $* + * = 0$

How does $*$ compare to the numbers we know? By definition, fuzzy games are confused with 0. Is $* < 1$? Yes! To check this, observe that Figure 24 is negative. In fact, if x is any positive number, $*+x$ is positive while $*+(-x)$ is negative.

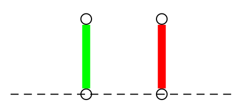


FIGURE 24. $* < 1$

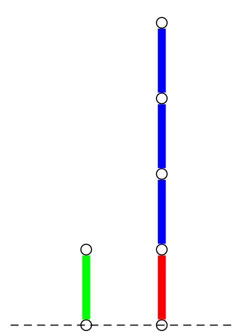


FIGURE 25. $* < \frac{1}{8}$

Figure 25 demonstrates that $* < \frac{1}{8}$ and generalizing to $* < \frac{1}{2^n}$ is straightforward. Left could only win if it is Left to move when only the green stick remains. But Right will simply take the green stick at his earliest opportunity, winning easily.

Example Let $G = \{0|*\}$ [1, p. 64]. It is easy to see that G is positive, since a move to 0 is winning and a move to a fuzzy game is losing. How does it compare to the positive numbers we know? We can easily verify that $G < 1$ by playing $G + (-1)$. The bracket notation can be seen as a board, just as a chessboard or a Hackenbush position would be. We can see that in the sum, $\{0|*\} + \{\ \ |0\}$ the options can be summarized as $\{-1|*+(-1), G\}$, which is clearly good for Right.[8] Similarly, it can be shown that for any positive number $x, G + (-x)$ is negative since the best Left can hope for would be $\{-x|*, -x\}$, which is negative. This is the first positive game with this property that we have encountered, so it deserves a name. Let $G = \uparrow$. We will denote $(-G) = \{*|0\}$ by $\downarrow$.

Example Let $G = \uparrow + *$. We will often abbreviate this as $\uparrow *$. By definition, we have $G = \{*, \uparrow | * + *, \uparrow\} = \{*, \uparrow |0, \uparrow\}$. G is certainly fuzzy: Left can move to a positive game, while Right can move to 0. We only know one fuzzy game—does $G = *$? No, since $\uparrow ** = \uparrow \neq 0$.

Example Let $G = \{0| \uparrow\}$. G is certainly positive, and it can easily be shown that G is less than all the positive Surreal Numbers. Perhaps $G = \uparrow$? No, since Right's move to $\uparrow + \downarrow = 0$ is winning in the sum $\{0| \uparrow\} + \downarrow$. Maybe $G = \uparrow + \uparrow$? The reader has enough tools to deduce the value of G in terms of games we have already defined, and it is left as an exercise. However, we would like to be able to identify games using techniques other than guess and check. In the next section, we consider three useful tools for formal manipulation.

2.8. Simplifying Games: Dominated moves, Reversible moves, and Gift Horses.

In studying any algebraic object, it is extremely useful to develop methods of simplification. We would on occasion like to verify that $G = H$ without playing the game $G + (-H)$. In this section, we will evaluate $\{0| \uparrow\}$ without guess and check. We begin this pursuit with a simple proposition:

Proposition 24. *If $G \in \mathbb{E}$ and A is a Left option from G, then Left can win by moving first in $G + (-A)$.*

Proof. Left plays $G \mapsto A$ and wins by Copycat. □

Definition 25. *Let $G = \{G^L|G^R\}$. A left option $X \in G^L$ is* **dominated** *if there exists $Y \in G^L$ such that $Y \geq X$.*

In chess, this is what we would call a "bad move." The following proposition allows us to ignore bad moves.

Proposition 26. *If $G \in \mathbb{E}$, removing a dominated option does not affect the equivalence class of G.* [1, p. 60-62]

[8] Right's option to G is **dominated**, as we shall see in Section 2.8.

Proof. Let $G = \{X, Y, ...|Z, ...\}$ and let X be dominated by Y. That is, $X \leq Y$. We want to show that $G+\{-Y, ...|-Z, ...\}$ is a zero game. Proposition 24 shows that the moves to $Y, Z, -Y, -Z$, etc. are losing (by Copycat). Suppose Left moves $G \mapsto X$. Then, Right wins by moving the total game to $X+(-Y)$ which is negative or zero. □

There is another class of moves which is not quite so bad. These moves are not so demonstrably wrong, but they fail to make forward progress. We call such moves reversible.

Definition 27. *A Left option G^L is* **reversible** *if from it there exists a Right option G^{LR} such that $G^{LR} \leq G$. We say the move G^L is reversible through G^{LR}.*

If Left plays a reversible move from G, Right would always like to reverse it, since this would make his position at least as good as G.

Proposition 28. *If $G \in \mathbb{E}$ and G^L is reversible through G^{LR}, G^L may be replaced by the left options from G^{LR} without affecting the equivalence class of G.* [1, p. 62-64]

Proof. Let $G = \{A, B, C, ...|D, E, F, ...\}$, and let Left's move to A be reversible through $A^R = \{U, V, W, ...|X, Y, Z, ...\}$. We'd like to show that $G + \{-D, -E, -F, ...|-U, -V, -W, -B, -C, ...\} = 0$. Left's moves to $B, C, ...$ or $-D, -E, -F, ...$ are countered by Right's moves to $-B, -C, ...$ or $D, E, F, ...$ and vice versa.

Suppose Left moves to A. Then, Right will respond in the same component, moving $A \mapsto A^R$, leaving Left to move in

$$A^R + \{-D, -E, -F, ...| - U, -V, -W, -B, -C, ...\},$$

or equivalently,

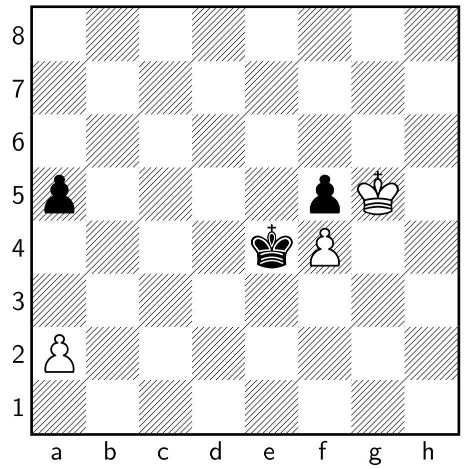
$$\{U, V, W, ...|X, Y, Z, ...\} + \{-D, -E, -F, ...| - U, -V, -W, -B, -C, ...\}.$$

Left's moves on A^R are easily countered. Consider, without loss of generality, his move to $-D$. This would leave Right with $A^R + (-D)$ which is at least as good for him as $G+(-D)$, which is winning by Proposition 24. Therefore Left's initial move to A is losing.

The last moves we must consider are Right's moves to $-U, -V, -W, ...$ Without loss of generality, let Right move to $-U$. This leaves $G + (-U)$ which is worse for Right than $A^R + (-D)$. Since Left could win the latter game by moving $A^R \mapsto U$, we see that Left can win $G + (-U)$ as well. This completes the proof, since we have shown that all first moves from $G + \{-D, -E, -F| - U, -V, -W, -B, -C, ...\}$ are losing.

□

Example Let G be the following game [3].



White's move **a2-a4** is winning, while Black's only move (**...a5-a4**) is losing. Therefore G is positive. We can write $G = \{0, *|*\}$. White's move to $*$ is reversible since from it Black can move to 0, and $0 < G$. Then, by Proposition 28, we can simplify and write $G = \{0|*\} = \uparrow$. This reversibility argument can be checked on the chessboard by observing that White can win this position even if he is not allowed to play **a2-a3** until after Black first plays **...a5-a4**.

Proposition 29 (Gift Horse Principle)**.** *Suppose* $G = \{G^L|G^R\}$ *and* $Y \lhd| G$*. Then,* $G = \{G^L, Y|G^R\}$. [1, p. 72]

Proof. Play $\{G^L, Y|G^R\} + \{-G^R| - G^L\}$. All moves except Left's move to Y lose trivially. But Left's move to Y leaves $Y + (-G)$ which is fuzzy or negative. □

We are now prepared to tackle the final example from Section 2.7, $G = \{0|\uparrow\}$. The diligent reader will have already found the following surprising result, the Upstart Equality [1, p. 71].

$$\boxed{\{0|\uparrow\} = \uparrow + \uparrow + * = \Uparrow *}$$

As usual, this identity could be checked by playing the sum. Instead we demonstrate how to simplify a game in the following proof.

Proof of the Upstart Equality. Let $G = \Uparrow *$. We can write

$$G = \{\uparrow *, \Uparrow \,|\, \Uparrow, \uparrow **\}.$$

After simplifying Right's latter move, we can see that it dominates the first, since $\uparrow < \Uparrow$. Then, by Proposition 26,

$$G = \{\uparrow *, \Uparrow \,|\, \uparrow\}.$$

G^R now looks as desired, but the LHS needs work. Left's move to $\Uparrow$ is reversible: Right would reply by moving to $\uparrow *$, and we have $\uparrow * < G$.

According to Proposition 28, we can replace Left's move to $\Uparrow$ by Left's options from $\uparrow *$. In particular, we have

$$\{\uparrow *, \uparrow, * | \uparrow\}.$$

We delete Left's move to $*$, as it is dominated by $\uparrow *$. Left's move to $\uparrow$ is reversible: Right moves to $*$ and we have $* < G$. By replacing $\uparrow$ by Left's options from $*$, we get

$$G = \{\uparrow *, 0 | \uparrow\}.$$

Left's move to $\uparrow *$ is also reversible. Right moves to $** = 0$ which is certainly better than G. Since Left has no options from 0, we get our claimed identity:

$$G = \{0 | \uparrow\}.$$

□

2.9. **Nim.** [1, p. 40-42] Recall that a Green Hackenbush edge may be re-

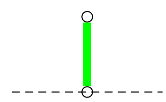


FIGURE 26. $*1 = * = \{0|0\}$

moved by either player. The Game of Nim is Hackenbush with only Green edges. Nim is a very simple Game, but it turns out to be very important to the study of **impartial games**[9]—games in which each player always has the same options. As a consequence of impartiality, Nim positions (or **Nimbers**) cannot be positive or negative. Then, all non-zero Nim positions are fuzzy. The simplest non-zero Nim position, shown in Figure 26, is $*1 = \{0|0\}$, which can be represented by a single Green edge. Don't confuse $*1$ with $1 + *$, which may be abbreviated $1*$. We will continue to call the position in Figure 26 $*$. The notation $*1$ is just meant to highlight its similarity with the other Nimbers, $*2$, $*3$, etc. In general, $*n$ is a string of n green edges. We can write

$$*n = \{0, *1, ... * n - 1 | 0, *1, ... * n - 1\}.$$

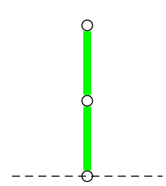


FIGURE 27. $*2 = \{0, * | 0, *\}$

[9]As opposed to **partizan games**.

We have already seen our first examples of Nim Addition with the identity $*+*=0$. It shouldn't be surprising that $*2+*2=0$ by Copycat; impartial positions are their own negatives.

Example

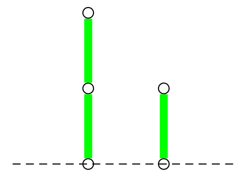


FIGURE 28. $G = * + *2$

How can we understand the game in Figure 28, $G = * + *2$? Figure 29 shows the options for each player.

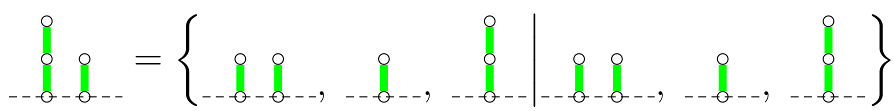


FIGURE 29. Options from Figure 28

In other words, $G = \{0, *, *2 | 0, *, *2\}$.

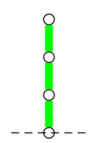


FIGURE 30. $*3$

Since this heap of 3 has the same set of options as G, we will write $G = *3$, and we can now write $*1 + *2 = *3$. Although you may take some comfort in this identity which so resembles the familiar $1 + 2 = 3$, you will have to get used to the fact that $*1 + *3 = *2$ and $*2 + *3 = *1 = *$. Later in this section we will prove the Minimal Excluded Rule which will allow us to evaluate general Nim positions, but first, let's get a sense of the strategy in simple Nim positions.

1-heap[10] games are simple: the first player takes the entire heap and wins. There are two types of 2-heap games. If the heaps are of equal size, Copycat shows the game is 0. If not, the first player can win by equalizing the size of the heaps.

[10] In Nim, strings of edges are often represented instead by heaps of blocks. 1-heap games are single string Hackenbush positions.

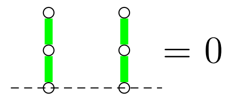


Figure 31

The 3-heap problem is more interesting. If any of the heaps are the same size, it's not really a 3-heap problem, so let's assume they're all different sizes. Eliminating an entire heap is a losing move, as is equalizing two heaps. Each of the identities above follow from observing that any move in the game $*1 + *2 + *3$ either eliminates a heap or equalizes two heaps.

Proposition 30 (Minimal-Excluded Rule)**.** *Let Left and Right have exactly the same options from $G \in \mathbb{E}$, all of which are Nim-heaps $*a, *b, *c, \ldots$ Then, $G = *m$, where m is the least number 0 or 1 or 2 or ... that is not among the numbers $a, b, c, \ldots$*

Proof. Let $*m$ be the so-called "Minimal Excluded" option of G (the **mex** of G). Play $G + *m$. It is not hard to see that all first moves are losing, since we have already explored the two heap problem. Any move on G would be losing as it would leave either two unequal heaps or a single heap. Any move on $*m$ is losing by Copycat. □

This result turns about to be far more important than it initially appears. At first sight, it appears to just be a tool to understand Nim. But it turns out that this observation is the key to the Sprague-Grundy Theorem, which reduces any impartial game to Nim! Proposition 30 is the inductive step of this theorem—we showed that if G is an impartial game and all of the options from G are nimbers, then G is also a nimber. To complete this inductive argument, we need a "base case." In fact, this base case is quite straightforward based on the formal construction of the class of games. Conway gives the following concise construction: "If L and R are any two sets of games, then there is a game $\{L|R\}$. All games are constructed in this way" [2, p. 15]. Then, the first game constructed (on Day 0, if you will) is $\{\emptyset|\emptyset\} = 0$. On Day 1, the games $1, -1$, and $*$ are constructed. Of the games so far, only 0 and $*$ are impartial. Since an impartial game must have impartial games for each of its options, the only new impartial game on Day 2 will be $*2$ (other new games include $\uparrow, 1/2$, and 2). Since Conway establishes that "all games are constructed in this way," the Inductive Hypothesis is justified.

3. Some Chess Ideas

From the outset, my goal in this thesis has been to consider a mathematical analysis of basic chess endgames. In particular, we will focus on two scenarios:

(1) Kings alone: The Opposition
(2) Winning Chances with a Lone Pawn (hereby, **KP vs. K**)

This approach differs significantly from that of Elkies. In particular, Elkies answers natural mathematical questions using a chessboard as the context. In this paper, however, I take the opposite perspective: I am considering questions which are natural from a chess perspective, and tackling them with the help of Combinatorial Game Theory. Ultimately, we will use CGT to understand the Opposition and KP vs. K without using any assumptions grounded in chess theory. However, in order to motivate this material, some chess background is helpful. This section contains only chess—no Mathematics. Any reader who knows how to play, but is not familiar with the concept of the Opposition in chess endgames is advised to read Section 3.1. This simple concept will come up constantly from now on. The reader who is more familiar with chess and interested in an instructive analysis of a practical Pawn Ending is advised to read Section 3.2.

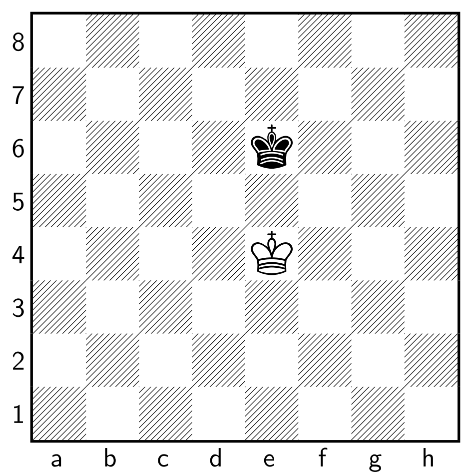


FIGURE 32. The Opposition: Nobody wants to move

3.1. **King versus King: The Opposition.** The critical position regarding the opposition is the apparently trivial and boring one shown in Figure 20. With no material on the board, a draw is the only possible outcome. Still, there is much to be said about this position. Most importantly, if Black is to move, White **has the Opposition**. Consider the following analogy to demonstrate two different uses of the opposition, one offensive and one defensive.

Suppose, rather than playing to checkmate, the Kings were playing football. The White King has the ball and wants to get to the Endzone (the 8th Rank). Since White has the advantage of the Opposition, he can do this. Black will have to choose to move left or right, and White will make a mad dash in the other direction.

1. . . ♔d6 2 ♔f5

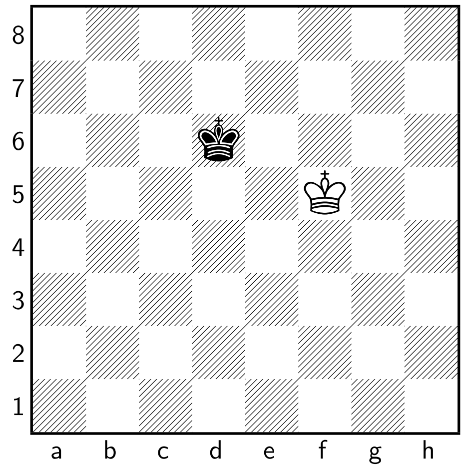


Figure 33. The offensive use of the Opposition

After Black responds with **2. . . ♔e7**, White can demonstrate a successful use of the Opposition by **3 ♔e5** (Figure 34), after which White has held on to the Opposition and made forward progress toward a touchdown.[11]

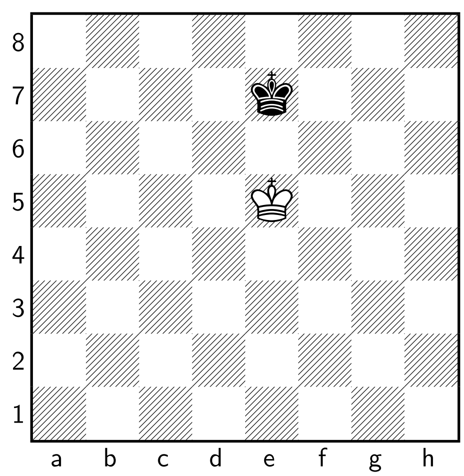


Figure 34. Walking Forward

Suppose, on the other hand, that back in Figure 32, Black had the Opposition (in other words, it was White's move). After any move, Black simply maintains the Opposition, and White can never make forward progress. This is the defensive use of the Opposition. The Opposition is the single most

[11]The fastest way to put 6 points on the board would be **3 ♔g6 ♔f8 4 ♔h7** with White's threat of **♔h8** unstoppable.

important topic in King and Pawn Endings, and will come up in each of the chess examples we consider.

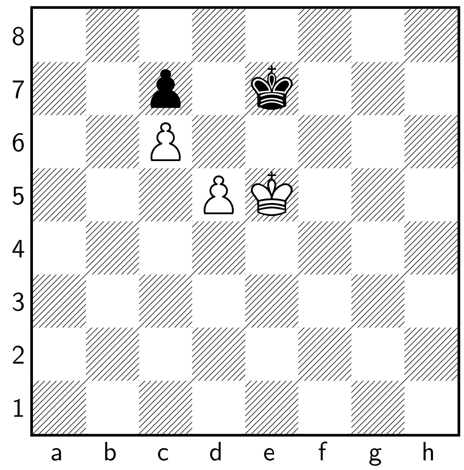


FIGURE 35. Triangulation

3.2. **Triangulation.** In this section we analyze the position shown in Figure 35. Black is in Zugzwang. That is, if it is Black to move, Black loses. This position is a terrific exercise to practice the Opposition which is very instructive. The reader is advised to set it up over a board and experiment to see if they can find the winning strategy for White after each of Black's options

(1) **. . . ♔e8**
(2) **. . . ♔d8**
(3) **. . . ♔f7** or **. . . ♔f8**

Notation for the solutions are listed below. After the solutions, there is a more intuitive explanation which can be understood without a chessboard, with the help of several diagrams. Still, a chessboard will makes things easier for the reader.

(1)

1	**. . .**	**♔e8**
2	**♔e6!**	**♔d8**

2. . . ♔f8 3 ♔d7 **[1-0]**

3	**♔f7!**	**♔c8**
4	**♔e7**	**♔b8**
5	**♔d7**	

[1-0]

(2)

1	**. . .**	**♔d8**
2	**♔f6!**	**♔e8**

2. . . ♔c8 3 ♔e7 **[1-0]**

3	**♔e6!**	

[**1-0**] We have now transposed to (1).

(3) We show . . . ♔f7 here, but this method wins against . . . ♔f8 as well (although against the latter move White could also win with the Opposition).

1	**. . .**	**♔f7**
2	**d6!**	**c×d6**

2. . . ♔e8 3 d×c7 [**1-0**]

3	**♔×d6**	**♔e8**
4	**c7**	

[**1-0**]

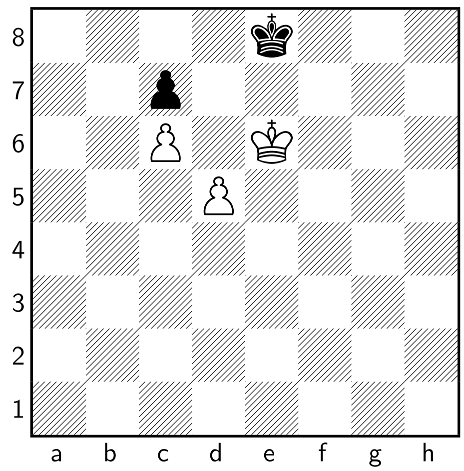


FIGURE 36. Taking the Opposition

The reader who has mastered the previous section should be able to easily demonstrate White's victory after **. . . ♔e8**. White will take the Opposition with **♔e6** (Figure 36) and then win with the "offensive" use of the Opposition (Figure 37). Here White seeks not to reach the endzone, but rather to capture the Black pawn.

After **. . . ♔d8**, White plays **♔f6!** taking the Diagonal Opposition (Figure 38). Now Black does not want to let the White King penetrate further, but will have no choice: White reaches the usual opposition by force.

Black's only other options are **. . . ♔f8** and **. . . ♔f7**, both of which are met by **d6**. Now White will promote one of his Pawns on the c8 square regardless of Black's response (Figure 39).

What if it is White's move? If White could pass, it would be an easy victory, since we just saw the Black is in Zugzwang. Fortunately, White *can* pass, by a method we call Triangulation. The name comes from the triangle formed by the squares **e5, f5,** and **e4**, as we shall see. White begins

1. . . ♔d8 2 ♔f7

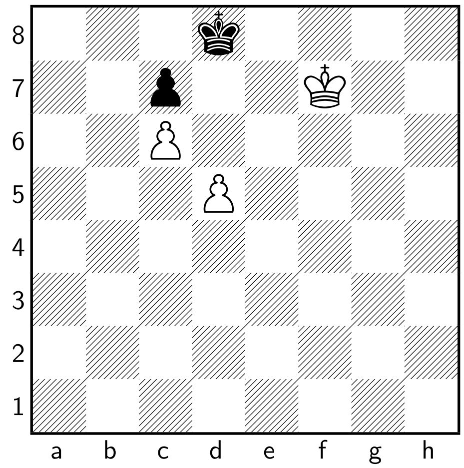


Figure 37. White will win the Pawn

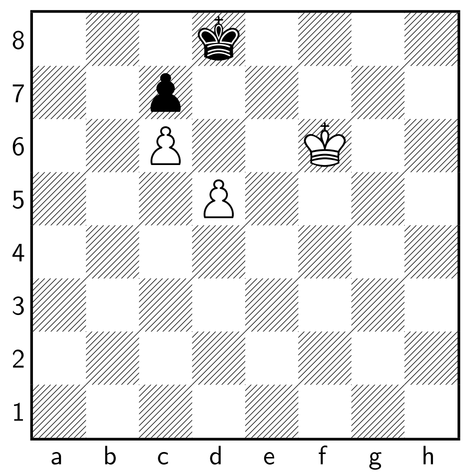


Figure 38. Diagonal Opposition

by moving **♔e5–f5**. Black cannot allow White to take the Opposition and cannot go to the f-file because he must stay close to the Pawns. Then, **. . . ♔d6** is the only option (Figure 40).

Now White defends his Pawn with **♔e4** (the second side of the triangle), leaving Black with an unhappy decision. **. . . ♔c5** will leave the Black King too far from the action as White marches forward to take Black's sole pawn. But on the other hand, after **. . . ♔e7**, White responds with **♔e5** reaching our original position with Black to move (Figure 35)! This technique has allowed White to effectively pass by making three moves while Black has

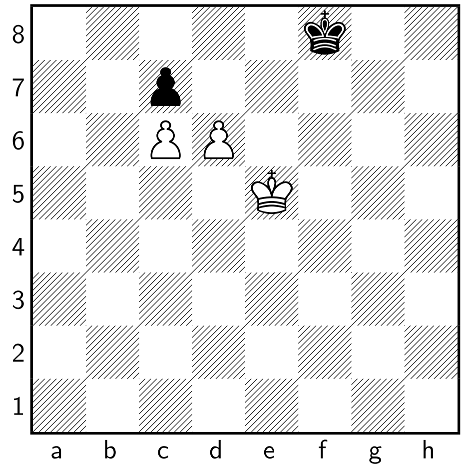


FIGURE 39. White will promote

1 ♔f5 ♔d6

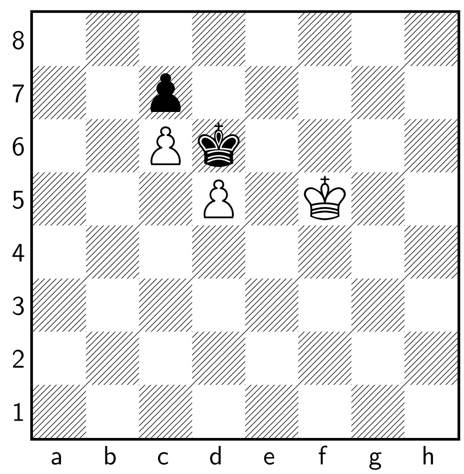


FIGURE 40. Beginning to Triangulate

only made two. White's triangular path leaves Black in Zugzwang, so White wins.

4. **Loopy games**

Although the position in Section 4.1 may be my first new contribution to the CGT world, Elkies deserves some credit here. He gives an analogous discussion of Triangulation, finding **tis** and **tis'n** instead of **on** [3]. Aside from that brief chess discussion, most of this section should be familiar to anyone who has read Chapter 11 of Winning Ways, but the presentation follows much more loosely than the sections above. This is primarily because I am not familiar with all of the relevant concepts Winning Ways uses. I took the definitions I needed from Winning Ways and worked out most of the proofs myself. I've included references to the relevant pages of Winning Ways, but I was unaware (or perhaps more accurately, willfully ignorant) of much of this material when writing this section. I still do not understand precisely the coverage of Section 4.2 given in Winning Ways, and I am unaware of Winning Ways or any other text discussing the loopy generalizations of dominated moves, reversible moves, and Gift Horses in Section 4.3.

4.1. **A Couple of Stoppers and a Loopy game.**

We have thus far considered only $\mathbb{E}$, the set of games which satisfy the Strong Ending Condition. Things are not as clean if we only assume the Weak Ending Condition. We will call these games **Stoppers**, $\mathbb{S}$. As we saw on page 8, the sum of two Stoppers may not be a Stopper. It follows that a game may not have an additive inverse, which prevents the use of much of our theory from Section 2. In this section, we study Stoppers, $\mathbb{S}$, and loopy games, $\mathbb{L}$. By construction, we have $\mathbb{E} \subset \mathbb{S} \subset \mathbb{L}$.

Definition 31. *A game is* **loopy** *if it does not satisfy the Weak Ending Condition.*

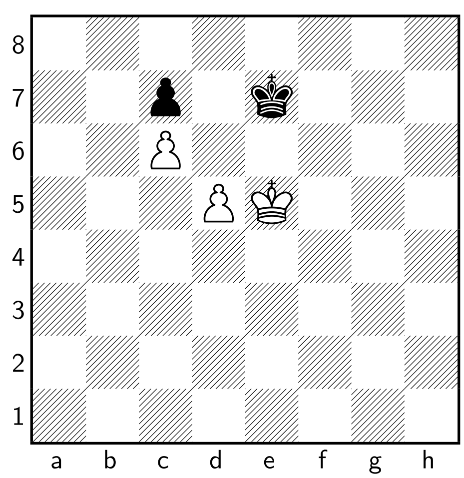


FIGURE 41. $\triangle$

Our first Stopper will be the game considered in Section 3.2. We will use the symbol $\triangle$ to represent the game in Figure 41 (this is also Figure 35).

How can we describe $\triangle$ using the language of CGT instead of chess notation? Recall that Black is in Zugzwang—that is, Black loses if he must move first. As with the trébuchet, we will disallow Black options from $\triangle$. And to make things simple, if it is White's move, we will disallow any strategy other than the method of "passing" described at the end of Section 3.2 (Triangulation). Then, we have $\triangle = \{\triangle | \ \}$. It is not hard to see that $\triangle$ is a Stopper. There is no infinite, alternating sequence, since there is not even a single legal move for Right. On the other hand, there certainly is an infinite play without alternating, since Left can "pass" indefinitely. $\triangle$ is certainly positive, but how big is it? In $\triangle + (-1)$ or indeed, $\triangle + x$ for any finite x, White (or Left) is still certainly winning. The ability to pass forever will outweigh any finite Hackenbush string. Even $\triangle + (-\omega)$ and $\triangle + (-\omega^\omega)$ are positive.

Proposition 32. *If $G \in \mathbb{E}$, $\triangle + G$ is positive.* [1, p. 338]

Proof. On each of Left's turns, Left will move $\triangle \mapsto \triangle$. Since Right cannot move on $\triangle$, Right's only moves will be to $\triangle + G^R, \triangle + G^{RR}, ...$ Since $G \in \mathbb{E}$, this sequence of Right moves must terminate. □

Proposition 32 shows that $\triangle$ is, in some sense, "larger" than any Ender, but unfortunately, we cannot write $G < \triangle$, since we have only defined $G < H$ when both G and H are Enders. Why did we impose this restriction? In order to write $G < \triangle$, we would need to check that $G + (-\triangle)$ is negative, but $-\triangle$ may not even exist. Then, we certainly must impose that $H \in \mathbb{E}$ (or at least, that $-H$ exists) in order to write $G < H$. The definitions in Section 2.2 also require that G is an Ender to ensure the nice property that $G < H$ implies $H > G$. We will delay making Loopy comparisons until we have suitable definitions.

The notation in this section is suggestive of the chess term triangulation, and was the symbol I used in my analysis of this position. From now on, we will revert to the usual language: $\triangle = \mathbf{on}$. This name is a slight variation of **On**, the ordinal numbers—since we have seen $\mathbf{On} \subset \mathbb{E}$, **on** is "larger" than any element of **On**. We will use the standard **off** for the role reversed version of **on**, and write

$$\mathbf{on} + \mathbf{off} = \mathbf{dud}.$$

where **dud** stands for **D**eathless **U**niversal **D**raw. **dud** is our first loopy game [1, p. 337].

4.2. **Loopy Comparisons.** As with the theory of Enders, the theory of loopy games begins with an understanding of the outcome of a loopy game. Theorem 5 gave a clear division of Enders into four Outcome Classes. Such a division is not possible for loopy games, as we have already seen. In particular, the proof of Theorem 5 is based on a contradiction with the Ending Condition.

A finite play[12] in a loopy game, γ, is easy to understand—the person who made the final moves wins as usual. But what if play is infinite: does this count as a win for Left? A win for Right? A draw? We will write $\gamma^\bullet$ to mean one possible rule for deciding the outcome of infinite plays in γ. In particular, the two most natural examples of $\gamma^\bullet$ are γ^+, in which infinite plays count as a win for Left, and the analogous γ^-. By infinite play, we refer to a countable sequence $G \mapsto G^L \mapsto G^{LL} \mapsto G^{LLR} \mapsto ...$ We will not consider uncountable play.

Often, there is a natural choice for $\gamma^\bullet$. In particular, if Black is down material, he would be quite happy with a draw (at least in the positions we will consider here). Therefore, it is natural to treat infinite plays as a "win" for Black, even though they would be a draw on the chessboard.

Definition 33. *A loopy game $\gamma^\bullet$ is* **fixed** *if no infinite play is drawn.* [1, p. 335]

Given fixed games $\alpha^\bullet$ and $\beta^\bullet$, what is the outcome of $\alpha^\bullet + \beta^\bullet$?

Example Consider the game **dud**$^+$+**dud**$^+$. Any play in this game must be infinite, and therefore at least one of the components must have an infinite play. Regardless, Left is winning in both components, so it is natural to call this game positive.

Example Consider the game **dud**$^+$+**dud**$^-$. If the play is infinite in only one component things are clear. But what if both component plays are infinite? Then, we say the sum is drawn.

Left wins a sum of loopy games only if he wins all components. A game is drawn if any component is drawn or if each player wins at least one component. [1, p. 335]

Definition 34. *If α is a game, $(-\alpha)$ is the role-reversed version of α.*

Observe that $\alpha + (-\alpha)$ is either 0 or **dud** and that this agrees with Definition 11 if $\alpha \in \mathbb{E}$.

Definition 35. *If $\alpha^\bullet$ and $\beta^\bullet$ are fixed loopy games, $\alpha^\bullet \geq \beta^\bullet$ if Left has a winning or drawing strategy in the combined games $\alpha^\bullet + (-\beta^\bullet)$, provided Right starts. Similarly, $\alpha^\bullet \leq \beta^\bullet$ if Right has a winning strategy or drawing strategy in $\alpha^\bullet + (-\beta^\bullet)$, provided Left starts.* [1, p. 348]

Proposition 36. $\alpha^\bullet \leq \beta^\bullet \iff \beta^\bullet \geq \alpha^\bullet$

Proof. Suppose Right has a winning strategy or drawing strategy in $\alpha^\bullet + (-\beta^\bullet)$, provided Left starts. Reversing the role of Left and Right demonstrates that Left has a winning or drawing strategy in $(-\alpha^\bullet) + \beta^\bullet$. The reverse direction works identically. □

Theorem 37. *$\leq$ is a partial order on $\mathbb{L}^\bullet = \{\gamma^\bullet \in \mathbb{L} : \gamma^\bullet$ is fixed$\}$.*

[12] Recall that the only way a play terminates is if there is no legal move for the player at-move. That is to say, a finite play refers to a play until the end of a game, not just the beginning of an infinite play.

Proof. $\leq$ is reflexive by Copycat. $\leq$ is anti-symmetric by definition: we will write $\alpha^\bullet = \beta^\bullet$ if $\alpha^\bullet \leq \beta^\bullet$ and $\beta^\bullet \leq \alpha^\bullet$. Transitivity is a little harder. Let $\alpha \leq \beta \leq \gamma$ be fixed loopy games. We'd like to show that Right can survive in $\alpha + (-\gamma)$ if Left starts. We first construct the strategy, and then verify that it works. Let Left move $\alpha \mapsto \alpha^L$, without loss of generality. Since $\alpha \leq \beta$, Right could, hypothetically, find a move which wins or draws in $\alpha^L - \beta$. If Right can win or draw by moving $\alpha^L \mapsto \alpha^{LR}$ in this hypothetical game, he will do the same in the real game. If not, then Right can win or draw this hypothetical game by moving $(-\beta) \mapsto (-\beta)^R$. Right now imagines a second hypothetical game, $\beta + (-\gamma)$. Right asks himself, "What would I do if Left moved $\beta \mapsto \beta^L$ (where β^L is just a role reversal of $(-\beta)^R$) in this new hypothetical game?" If Right can win or draw by making a move on the $(-\gamma)$ component, he will make the same move in the real game. If not, then Right can win or draw by moving $\beta^L \mapsto \beta^{LR}$. We now return to the first hypothetical game. It is clear that Right can survive in the game $\alpha^L + (-\beta^{RL})$—Left has just made a single move since the last time we looked at it. As above, if Right can win by moving on the α component, he will do the same in the game. If not, he will choose his move on the hypothetical β component and repeat this process. Either this process will produce a move for Right, or it will produce an infinite play in β which is winning or drawing for both Left and Right. But β is fixed, so this is a contradiction. Then, this strategy always finds at least one viable move in the real game.

We now show that if Right uses this strategy, he cannot lose. Suppose Right uses this strategy. The play in $\alpha + (-\gamma)$ induces two associated plays, one in $\alpha + (-\beta)$ and one in $\beta + (-\gamma)$, using the thought process above. Right wins or draws both of these plays by construction. If the total play in $\alpha + (-\gamma)$ is finite, Right wins because he must have made the last move. Suppose the total play is infinite and that Left wins $\alpha + (-\gamma)$. If both components are infinite, then the outcome of the induced plays suggest that Right must be winning on both of the β components. But these plays are exactly opposite of each other. Finally, suppose that only one component has infinite play (say α). Since Left is winning this component, the induced play in $(-\beta)$ must be infinite favor Right. But then the induced play in β is infinite and favors Left, which implies that the play in $(-\gamma)$ is infinite. But this is a contradiction. Then, Left cannot win against this strategy.[13] □

In proving Theorem 21, transitivity was straight-forward because we could add $B + (-B)$ without any problems. In the proof above, we have to resort to a method which is reminiscent of an old trick allowing anyone to perform well against two Grandmasters simultaneously. Imagine you play Black

[13] An extremely similar argument can be found on pages 349-359 of Winning Ways, but I was not aware of this argument until two days before the deadline for this paper. In fact, I still don't fully understand the argument there; it relies on concepts such as the **sign** of a game and then REMAIN-ON TOP condition which I am not familiar with and are not discussed in this paper.

against the World Champion, Vishy Anand, and White against the world's highest rated player, Magnus Carlsen. When Anand moves, you play that move against Carlsen. When Carlsen responds, you play that move against Anand. With this thoughtless play, you guarantee yourself an even score.

Theorem 38. *$=$ is an equivalence relation on $\mathbb{L}^\bullet$.*

Proof. As with $\leq$, $=$ is reflexive by Copycat. $=$ is symmetric by construction. Transitivity follows immediately from transitivity of $\leq$. □

Theorem 17 tells us that zero games in $\mathbb{E}$ equal 0. We now generalize:

Theorem 39. *$G^\bullet \in \mathbb{L}^\bullet$ is a zero game $\iff$ $G^\bullet = 0$.*

Proof. The reverse direction is immediate, since $0 \in \mathbb{L}$ is the same 0 from $\mathbb{E}$. For the forward direction, we just need to see that $G^\bullet + 0$ and $-(G^\bullet) + 0$ are zero games. But this is not hard, since the only legal moves are in $G^\bullet$ and $-(G^\bullet)$, respectively. □

4.3. **Simplifying Loopy games.** In this section we generalize the propositions from Section 2.8.

Proposition 40 (Generalization of Proposition 26)**.** *If $G \in \mathbb{L}^\bullet$, removing a dominated option does not affect the equivalence class.*

Proof. Let $G^\bullet = \{X^\bullet, Y^\bullet, ...|Z^\bullet, ...\}$ be a fixed loopy game and let $X^\bullet$ be dominated by $Y^\bullet$. That is, $X^\bullet \leq Y^\bullet$. Let $H^\bullet = \{Y^\bullet, ...|Z^\bullet, ...\}$. We first show that $G^\bullet \leq H^\bullet$. Suppose Left starts in the combined game $G^\bullet - H^\bullet$. If Left moves $G^\bullet \mapsto X^\bullet$, Right moves $-H^\bullet \mapsto -Y^\bullet$. By assumption, Right can win or draw this game provided Left starts. All of Left's other moves are easily countered by Copycat.

We still need to see that $H^\bullet \leq G^\bullet$. Suppose Left starts in $H^\bullet + (-G^\bullet)$. All moves are countered by Copycat. □

Proposition 41 (Generalization of Proposition 28)**.** *Let $G^\bullet \in \mathbb{L}^\bullet$ and let $G^{\bullet L}$ be reversible through $G^{\bullet LR}$. Then, the option $G^{\bullet L}$ can be replaced with the Left options from $G^{\bullet LR}$ without affecting the equivalence class of G.*

Proof. Let $G^\bullet = \{A^\bullet, B^\bullet, C^\bullet, ...|D^\bullet, E^\bullet, F^\bullet, ...\}$ be a fixed loopy game and let $A^\bullet$ be reversible through $A^{\bullet R} = \{U^\bullet, V^\bullet, W^\bullet, ...|X^\bullet, Y^\bullet, Z^\bullet, ...\}$. Let $G^\bullet = H^\bullet$ where $H^\bullet = \{U^\bullet, V^\bullet, ..., B^\bullet, ...|D^\bullet, E^\bullet, F^\bullet\}$. We'd like to show $G = H$. Consider the game

$$G^\bullet - H^\bullet = \{A^\bullet, B^\bullet, ...|D^\bullet, ...\} + \{-D^\bullet, ...| - U^\bullet, ... - B^\bullet, ...\}.$$

All of Left's options except $G^\bullet \mapsto A^\bullet$ are met by Copycat. If Left moves to $A^\bullet$, Right moves to $A^{\bullet R}$ leaving Left to move in

$$\{U^\bullet, V^\bullet, W^\bullet, ...|X^\bullet, Y^\bullet, Z^\bullet, ...\} + \{-D^\bullet, ...| - U^\bullet, ... - B^\bullet, ...\}.$$

Now Left's options on the first component are met by Copycat. If Left moves $-H^\bullet \mapsto -D^\bullet$ (without loss of generality) we find $A^{\bullet R} + (-D^\bullet)$. By assumption, this is at least as good for Right as $G^{\bullet}+(-D^{\bullet})$, which is winning by Copycat. Therefore, $G^{\bullet} \leq H^{\bullet}$. Now consider the game

$$H^{\bullet} - G^{\bullet} = \{U^{\bullet}, V^{\bullet}, ..., B^{\bullet}, ...|D^{\bullet}, E^{\bullet}, F^{\bullet}\} + \{-D^{\bullet}, ...| - A^{\bullet}, -B^{\bullet}, ...\}.$$

All moves other than $H^{\bullet} \mapsto U^{\bullet}$ (without loss of generality) are met by Copycat. Suppose Left moves the total game to $U^{\bullet} - G^{\bullet}$. This is no better for Left than $U^{\bullet} - A^{\bullet R}$, which Right could draw by Copycat after moving $-A^{\bullet R} \mapsto -U^{\bullet}$. □

We shall see examples in which all possible moves (say, for Left) in a loopy game are reversible, only to be replaced with more reversible options. In this case, we can replace G^L with $\emptyset$. The proof of the Gift Horse Principle applies to loopy games as written.

5. Chess endgames as mathematical games

5.1. **Opposition as a Loopy game.** In this section, we give a mathematical analysis of the Opposition we considered previously in Section 3.1.

5.1.1. *The 3x3 Game.* Suppose we constrain both Kings to stay within the 3x3 square a6-c8, and suppose that White wants to reach either a7, b7, or c7 (marked with crosses). If the White King reaches one of these key squares, we will say that White wins this component and treat it as **on**. This convention will mean that the option of reaching a key square will dominate all other moves. This is easy to see since in any combined game, reaching a key square guarantees a win or draw. Similarly, any move which allows the opponent to reach a key square is reversible through **on**. Then, we can safely ignore any such move. We will consider all infinite plays in which White does not reach a key square as a win for Black. In this section we will drop the $^{\bullet}$ notation. We will use blue crosses to mean White is attacking (so infinite plays are good for Black) and red crosses to mean the opposite.

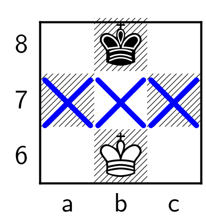


FIGURE 42. The Opposition

Let **Figure 42** be called G. Black's moves from G are clearly losing. We saw in Section 3.1 that if White is to move, Black wins by the "Defensive Use" of the opposition. Then $G = 0$ since it is a zero game. If we move the

Kings but maintain Direct Opposition, we still find 0. There are four other positions (up to symmetry) which we need to consider in this 3x3 Opposition Game.

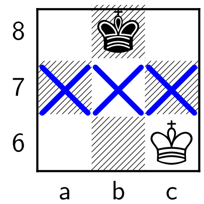


FIGURE 43. $*$

Figure 43 is formally equivalent to $\{0|0, {}_{c6}O_{a8}, {}_{c6}O_{a7}\}$, where the subscripts on the latter two options refer to the position of each King. Black's move to a8 and a7 are both reversible since they allow White to reach c7. Since Black has no options from **on**, they can be eliminated as options (in other words, they can be replaced with the empty set). Thus, we find that ${}_{c6}O_{b8} = *$.

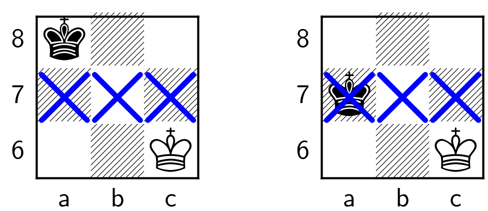


FIGURE 44. **on**$*$

Figure 44 The two games in Figure 44 can both be written as $\{\textbf{on}|*\}$ since in each of them, all other options are dominated or reversible. We will abbreviate this notation to **on**$*$, as this game will come up frequently. It is not hard to see that **on**$*$ is greater than $\uparrow$,[14] but confused with all positive numbers, $*$, and higher multiples of $\uparrow$. We will write $-\textbf{on}* = *\textbf{off}$. The similar games $\{\textbf{off}|*\}$ and $\{*|\textbf{on}\}$ do not merit their own names since they are both equal to 0.

Figure 45 The final position we'll consider in this 3x3 game is the fuzzy game shown in Figure 45 which we can write as $\{\textbf{on}|0\}$. Using **Zip** to suggest 0, we will abbreviate this game to **on-zip**. Observe that it is impossible for any game to be strictly greater than either **on-zip** or **on**$*$, since Left will

[14] Recall that $\uparrow *$ is fuzzy.

always be able to draw if allowed to move first. **on-zip** is greater than $*$ and $\downarrow$ but confused with all non-negative numbers and all multiples of $\uparrow$. Naturally, **zip-off** is the negative of **on-zip**. We will not use the names **off-zip**, and **zip-on** since -1 and 1 are more natural. All other positions

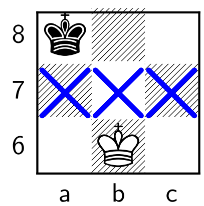


FIGURE 45. **on − zip**

(again, up to symmetry) in the 3x3 Game are equivalent to **on**.

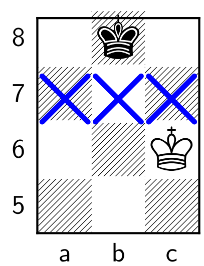


FIGURE 46. $*$

5.1.2. *The 3x4 Game.* Now that we understand these 3x3 positions, let's consider the 3x4 Game, where White's King may now be as far back as the 5th rank. Our analysis of the 3x3 case extends neatly. If the Kings are in opposition, we still find 0.

Figures 46 and 47 If we place the 3x3 position for $*$ on the 3x4 board, as in Figure 46, we find the same value. To show that Figure 46 equals $*$, consider the combined game Figure 46+$*$. Neither player wants to move $* \mapsto 0$ until the other player takes the Opposition. Similarly, neither players wants to take the Opposition before the other player moves $* \mapsto 0$. Black's only non-dominated move in the chess component is **. . . ♔c8** after which White moves on $*$ and wins. White's move **♔b6** is similarly losing, as it takes the Opposition. White's move **♔c5** loses to **. . . ♔b7** and **1 ♔b5** is reversible through **1. . . ♔a7**, since White's only viable response **2 ♔c6** only leads to a repetition, which favors Black. Nothing changes if we translate the King positions, as long as White does not have the immediate option of

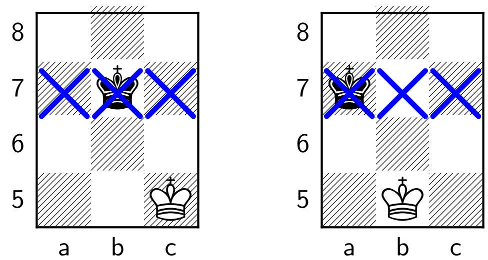


FIGURE 47. More $*$'s

moving to a key square and the relative position of the Kings is preserved; both of the games in Figure 47 are equal to $*$, since each of the games in Figure 47 give zero games when combined with $*$. The positions we know as $\mathbf{on} - \mathbf{zip}$ and $\mathbf{on}*$ from the 3x3 Game have the same values on this larger board, since White's new options are dominated by **on**.

Figure 48 shows ${}_{b5}O_{a8}$. A priori, we find

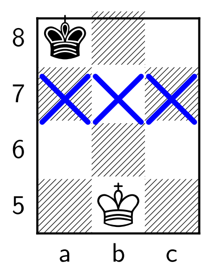


FIGURE 48. A more complicated game

$$ {}_{b5}O_{a8} = \{0, {}_{a5}O_{a8}, {}_{c5}O_{a8}, \mathbf{on} - \mathbf{zip}, \mathbf{on}* | 0, *, {}_{b5}O_{b8}\}. $$

We already saw that $\mathbf{on}* > 0$, so Left's option to 0 is dominated. We will show that Left's moves to ${}_{a5}O_{a8}$ and ${}_{c5}O_{a8}$ are dominated.

Left's option to ${}_{a5}O_{a8}$ is dominated by $\mathbf{on}*$ since ${}_{a5}O_{a8} \leq \mathbf{on}*$. To check this, we show that if Left starts, Right can win or draw in the combined game:

$$ \{0, {}_{b5}O_{a8}, \mathbf{on} - \mathbf{zip} | 0, *, {}_{a5}O_{b8}\} + *\mathbf{off}. $$

If Left wants to win, Right can not be allowed to reach **off**. Left's only option which prevents this leaves ${}_{a5}O_{a8} + *$, which Right wins by moving the first component to **on-zip**.

Left's option to ${}_{c5}O_{a8}$ is also dominated by $\mathbf{on}*$ since ${}_{c5}O_{a8} \leq \mathbf{on}*$. To check this, we show that if Left starts, Right can win or draw in the combined game:

$$_{a5}O_{a8} - \mathbf{on}* = \{\mathbf{on}*, {}_{b5}O_{a8}, \mathbf{on} - \mathbf{zip} | 0, *, {}_{a5}O_{b8}\} + *\mathbf{off}.$$

As above, Left's only viable move to play for a win would leave $_{c5}O_{a8} + *$ which Right wins. We can now rewrite our expression for $_{b5}O_{a8}$ in the following, more understandable form:

$$_{b5}O_{a8} = \{\mathbf{on}*, \mathbf{on} - \mathbf{zip} | 0, *, {}_{b5}O_{b8}\}.$$

Figure 49 shows $_{b5}O_{b8}$, the only unfamiliar option from the game above. We can write this game as

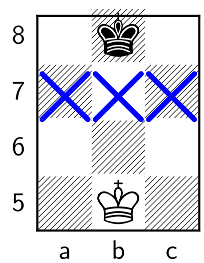


FIGURE 49. Another complicated game...

$$_{b5}O_{b8} = \{0, *, {}_{a5}O_{b8} | 0, *, {}_{b5}O_{a8}\}.$$

Black's move to $_{b5}O_{a8}$ gives White the option of forcing him to move to either $*$ or 0, so this option does not provide him any value. More precisely, his move to $_{b5}O_{a8}$ is reversible. We can similarly find that regardless of the White King's position on the 5th rank, Black's move from b8 to a8 is always reversible. In particular, we find that

$$_{b5}O_{b8} = \{0, *, {}_{a5}O_{b8} | 0, *\}$$

and

$$_{a5}O_{b8} = \{0, *, {}_{b5}O_{b8} | 0, *\}.$$

In fact, these two games are equal. In the combined game

$$_{b5}O_{b8} - {}_{a5}O_{b8} = \{0, *, {}_{a5}O_{b8} | 0, *\} + \{0, * | 0, *, -{}_{b5}O_{b8}\},$$

the moves to 0 or $*$ are obviously losing. Then, the only viable move for the first player is to toggle his King on the "attacking board." The second player will respond by toggling his own "attacking King." No progress will be made unless one of the Kings vacates his defensive stronghold, so play will be infinite on both components. Since Left wins one component and Rights wins the other, we find the second player can easily ensure a draw. We will call each of these games $\mathbf{on}_{*2}$. It is not hard to see that $\mathbf{on}_{*2}$ is confused with 0 and $*$, but greater than all other Nimbers. We can now further understand the position considered above when the Black King is off-center:

$$_{b5}O_{a8} = \{\mathbf{on}*, \mathbf{on} - \mathbf{zip} | 0, *, \mathbf{on}_{*2}\}.$$

There are three other positions with the Black King off-center and the White King on the 5th rank, but they are now straight-forward given our analysis of $_{b5}O_{a8}$ above.

$$_{a5}O_{a8} = \{0, \mathbf{on} - \mathbf{zip}, {}_{b5}O_{a8} | 0, *, \mathbf{on}_{*2}\}$$

$$_{c5}O_{a8} = \{\mathbf{on}*, \mathbf{on} - \mathbf{zip}, {}_{b5}O_{a8} | 0, *, \mathbf{on}_{*2}\}$$

$$_{c5}O_{a7} = \{\mathbf{on}*, * | *, \mathbf{on}_{*2}\}$$

The only part of these three lines which requires comment is the omission of Black's option to move to $_{c5}O_{a8}$ in the third line, but that option is dominated by $\mathbf{on}_{*2}$. This concludes our analysis of the Opposition, but it should not be so hard to extend this to a wider or longer board.

5.2. **King and Pawn vs. King.** In this section, we consider positions in which White has a King and a Pawn on the c-file, whereas Black has a lone King. The astute reader will notice that we have shifted our 3x3 board, because the edge has some stalemate implications which we avoid this way. As in Section 5.1, if White reaches a "key square," we will treat the position as **on** and say White wins the component. Here, however, the key squares are determined by the position of the Pawn. If Black is to move and is stalemated or can take the Pawn, we will say that Black can move to **off**. Infinite plays count as a win for Black.

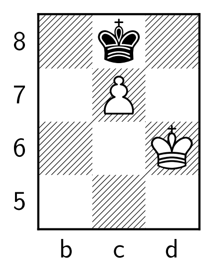


FIGURE 50. A zero game

5.2.1. *Pawn on the 7th.* Consider first positions in which the Pawn is on the 7th rank.

Figure 50 is clearly a zero game. White to move either stalemates or loses the Pawn, while Black to move must allow the White King to reach d7.

Figure 51 is equal to $\mathbf{zip} - \mathbf{off}$, since White's other moves are reversible through **off**.[15] If Black's King is off-center, we have check: a rather unset-

[15]Recall that if Black is stalemated or can take the Pawn we say he can move to **off**.

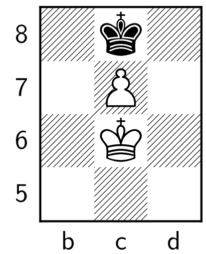


FIGURE 51. **zip − off**

tling situation in the world of mathematical games, but thankfully we can avoid this issue.[16]

5.2.2. *Pawn on the 6th.* We now complicate things by moving the Pawn back to the 6th rank.

Figure 52 is a zero game. White's option to move the King back loses to **. . . ♔c7** while pushing the Pawn only leads to stalemate. On the other hand, if Black starts **. . . ♔c8** loses.

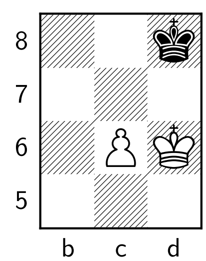


FIGURE 52. A zero game

In **Figure 53**, White's option to move the King back is dominated by **c6–c7**, while Black's option **. . . ♔b8** is dominated by **. . . ♔d8**. The only remaining option is for either player to move to 0, so Figure 53 equals $*$. Changing the Black King's position to b8 gives **on**$*$ since all other options are dominated. We have covered all positions with the White King and Pawn on the 6th.

Figure 54 shows a game we will call **blockade**. We have

$$\mathbf{blockade} = \{{}_{b5}\mathbf{6}_{c7} | {}_{c5}\mathbf{6}_{b8}, {}_{c5}\mathbf{6}_{c8}\}.$$

[16]WW explains the theory of impartial games with entailing moves. Here entailing moves are complicated by the partizan and loopy nature of the game.

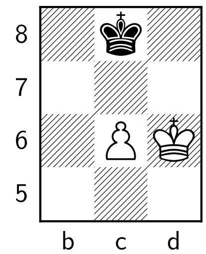


FIGURE 53. $*$

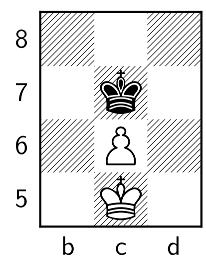


FIGURE 54. **blockade**: a negative game

This notation should be familiar from the previous section, with the subscripts denoting the position of each King. Here we use **6** to mean that the Pawn is on the 6th rank. Black's move **1. . . ♚b8** is reversible, since White's response **2 ♔d6** leaves a positive game, whereas the original position is negative. Then, we can replace the Right option ${}_{c5}\mathbf{6}_{b8}$ with $*$. Black's move **1. . . ♚c8** is also reversible, since White's response **2 ♔d6** leaves $*$, and $* \geq$ **blockade**. Then, we can replace the Right option ${}_{c5}\mathbf{6}_{c8}$ with 0. Finally, Left's option to ${}_{b5}\mathbf{6}_{c7}$ can be replaced by **blockade** since the two games are equal. To see this, suppose Right moves first in the difference **blockade** $- {}_{b5}\mathbf{6}_{c7}$. If Right moves his attacking King, Left will do the same. If Right continues this, play will be drawn since it will be infinite in both components. Right's only other try is to move the defending King, which we have seen is equivalent to moving to 0 or $*$. Then, Left will do the same in the other component, drawing by Copycat. If Left moves first, Right uses the same strategy, so the two games are equal. We have reduced Figure 54 to

$$\textbf{blockade} = \{\textbf{blockade} | 0, *\}.$$

blockade is less than $*$ and all multiples of $\downarrow$, greater than all negative numbers, and confused with both **on-zip** and **on**$*$.

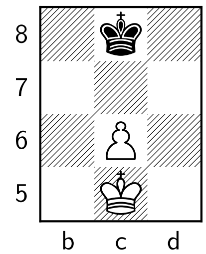


Figure 55. Another negative game

Figure 55 shows ${}_{c5}\mathbf{6}_{c8}$. The popular book Pandolfini's Endgame Course describes the defensive technique of toggling the King between c7 and c8 the **Frontal Defense**. We will adopt that name, referring to ${}_{c5}\mathbf{6}_{c8}$ as **FD** from now on.

$$\mathbf{FD} = \{*, {}_{b5}\mathbf{6}_{c8} | \mathbf{blockade}, {}_{c5}\mathbf{6}_{b8}\}.$$

White's move to **1 ♔b5** is reversible since **1. . . ♔c7** leaves **blockade**, and **blockade** $\leq$ **FD**. To check this, suppose Left moves first in

$$\mathbf{blockade} - \mathbf{FD} = \{\mathbf{blockade}|0, *\} + \{-\mathbf{blockade}, -{}_{c5}\mathbf{6}_{b8}|*, -{}_{b5}\mathbf{6}_{c8}\}.$$

Left's move on the first component is met by Right moving the second component to $*$. Left's move to $-{}_{c5}\mathbf{6}_{b8}$ allows Right to win on both components after moving that component to $*$**off**. If Left moves to $-$**blockade**, Right "passes" on that component and then draws by Copycat. This shows that the Left option ${}_{b5}\mathbf{6}_{c8}$ is reversible, so we replace it with **blockade**. But after making this replacement, this option is dominated by $*$.

The Black move **. . . ♔b8** is dominated by **. . . ♔c7**, since **blockade** $\leq$ ${}_{c5}\mathbf{6}_{b8}$. To check this, suppose Left moves first in

$$\mathbf{blockade} - {}_{c5}\mathbf{6}_{b8} = \{\mathbf{blockade}|0, *\} + \{-\mathbf{blockade}, -\mathbf{FD}|0, *\mathbf{off}, -{}_{b5}\mathbf{6}_{b8}, -{}_{d5}\mathbf{6}_{b7}\}$$

Left's move to **blockade** allows Right to move to $*$ and win. Left's move to $-$**blockade** allows Right to "pass" and win as above. Finally, his move to $-$**FD** is dominated by $-$**blockade**. Then, we can eliminate Black's option **blockade** from our evaluation of **FD**. We have reduced Figure 55 to

$$\mathbf{FD} = \{*|\mathbf{blockade}\}.$$

Given this form, it is not hard to see that **blockade¡FD**.

Figure 56 shows a third negative game, which we can write as

$${}_{b5}\mathbf{6}_{c8} = \{*, \mathbf{FD}|\mathbf{blockade}, {}_{b5}\mathbf{6}_{a8}, {}_{b5}\mathbf{6}_{d8}\}.$$

We now show that ${}_{b5}\mathbf{6}_{c8} = \mathbf{FD}$. We need to see that the second player can win or draw in the combined game

$${}_{b5}\mathbf{6}_{c8} - \mathbf{FD} = \{*, \mathbf{FD}|\mathbf{blockade}, {}_{b5}\mathbf{6}_{a8}, {}_{b5}\mathbf{6}_{d8}\} + \{-\mathbf{blockade}|*, -{}_{b5}\mathbf{6}_{c8}\}.$$

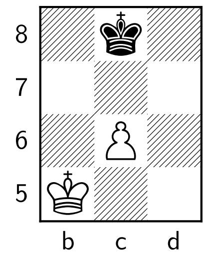


FIGURE 56. A third negative game

If Left moves to $*$, Right does the same and wins. If Left moves the first component to **FD**, Right draws by moving that component to **blockade** which we have already seen to be less than **FD**. If Left moves the second component to $-$**blockade**, Right moves the first component to **blockade** and draws by Copycat. Then, ${}_{b5}\mathbf{6}_{c8} \leq \mathbf{FD}$. If Right moves the first component to **blockade**, Left moves the second component to $-$**blockade** and draws by Copycat. Right's other two moves on the first component allow Left to win outright by moving the King forward on that component. Right's move to $*$ loses trivially, and his move to $-{}_{b5}\mathbf{6}_{c8}$ allows Left to draw by moving the first component to **FD**, since ${}_{b5}\mathbf{6}_{c8} \leq \mathbf{FD}$. We have shown that all first moves fail to win, so ${}_{b5}\mathbf{6}_{c8} = \mathbf{FD}$. There are three more positions to consider, all with the Black King on b8.

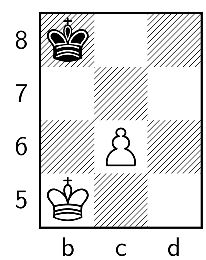


FIGURE 57. A fuzzy game

Figure 57 can be written as

$${}_{b5}\mathbf{6}_{b8} = \{0, {}_{c5}\mathbf{6}_{b8} | \mathbf{FD}, \mathbf{blockade}\}.$$

We have already compared Right's options and we can recognize the former option as dominated. White's move **♔c5** is reversible through **. . . ♚c7**, since **blockade** $\leq {}_{b5}\mathbf{6}_{b8}$. To check this, suppose Left moves first in the combined game:

$$\mathbf{blockade} - {}_{b5}\mathbf{6}_{b8} = \{\mathbf{blockade}|0, *\} + \{-\mathbf{blockade}|0, -{}_{c5}\mathbf{6}_{b8}\}.$$

If Left "passes" on the first component, Right will do the same on the latter component, moving the total game to $\mathbf{blockade} - {}_{c5}\mathbf{6}_{b8}$. If Left continues to pass on the first component, Right will draw by toggling back and forth; play will be infinite in both components. Otherwise, Left will eventually have to move the second component to $-\mathbf{blockade}$, after which Right will "pass" on that component and then draw by Copycat. Then, Left's option ${}_{c5}\mathbf{6}_{b8}$ can be replaced with **blockade**. But now this option is dominated by 0. Then, we can write Figure 57 as

$$ {}_{b5}\mathbf{6}_{b8} = \{0 | \mathbf{blockade}\}. $$

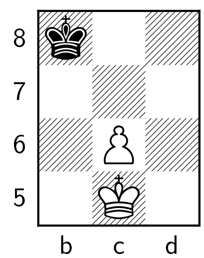


FIGURE 58. Another fuzzy game

Figure 58 can be written as

$$ {}_{c5}\mathbf{6}_{b8} = \{0, \mathbf{on}*, {}_{b5}\mathbf{6}_{b8}, {}_{d5}\mathbf{6}_{b8}, | \mathbf{FD}, \mathbf{blockade}\}. $$

Left's move to 0 is dominated. Left's move to ${}_{b5}\mathbf{6}_{b8}$ is dominated by ${}_{d5}\mathbf{6}_{b8}$. Right's move to **FD** is dominated. And Left's option to ${}_{d5}\mathbf{6}_{b8}$ is reversible through ${}_{d5}\mathbf{6}_{c7}$, since $\mathbf{blockade} \leq {}_{c5}\mathbf{6}_{b8}$. To check this, we need to see that Right can win or draw if Left starts in the combined game

$$ \mathbf{blockade} - {}_{c5}\mathbf{6}_{b8} = \{\mathbf{blockade} | 0, *\} + \{-\mathbf{blockade} | *\mathbf{off}, -{}_{d5}\mathbf{6}_{b8}\}. $$

If Left "passes" on the first component, Right will win by making the forcing move to $*\mathbf{off}$. If Left moves the second component to $-\mathbf{blockade}$, Right will pass and then draw by Copycat. This shows that Left's option to ${}_{d5}\mathbf{6}_{b8}$ is reversible and can be replaced with **blockade**. But this option is now dominated. Then, we can write Figure 58 as

$$ {}_{c5}\mathbf{6}_{b8} = \{\mathbf{on}* | \mathbf{blockade}\}. $$

Finally, **Figure 59** shows ${}_{d5}\mathbf{6}_{b8} = \{\mathbf{on}*, {}_{c5}\mathbf{6}_{b8} | {}_{d5}\mathbf{6}_{c7}, {}_{d5}\mathbf{6}_{c8}\}$. This game is easily seen to be equal to ${}_{c5}\mathbf{6}_{b8}$.

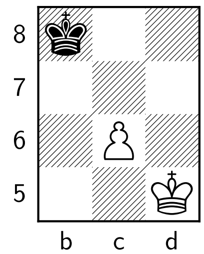


FIGURE 59. A third fuzzy game

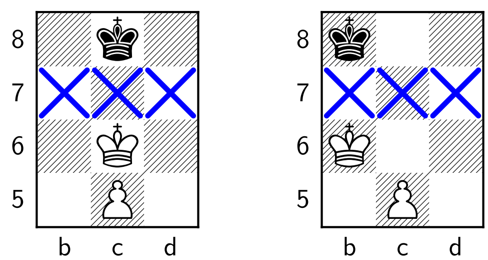


FIGURE 60. 1 and 1

5.2.3. *Pawn on the 5th.* We can now move the Pawn back to the 5th rank.

Figure 60 Consider the two positions in Figure 60, in which the Kings are in opposition, with the White King on the 6th rank. The chess player may recognize both of these games as winning for White regardless of whose move it is. This example may be somewhat surprising to some chess players, since normally the Opposition favors the second player, while this position favors White regardless of who moves. In the chess literature, this anomaly is called the **6$^{\text{th}}$ Rank Exception**. Both of these games are equal to 1. This is not hard to check.

Figure 61 Next we consider positions where the Kings are not in Opposition, as in Figure 61. We can see easily that this game is equal to $1 + *$, which is often abbreviated to $1*$. In the combined game $* - 1+$Figure 61, moving to $*$ would allow the second player to take the Opposition and win, taking the Opposition allows the second player to win by moving to $*$. Finally, Right's option to move $-1 \mapsto 0$ is never going anywhere, so there is no point in making it now.

Figure 62 As in earlier sections, we will say any position with the White King on the 7th rank is **on**. Then, it is obvious that the two positions in Figure 62 are equal to $\{\mathbf{on}|1*\}$ and $\{\mathbf{on}|1\}$ respectively, which will be abbreviated as the caption shows.

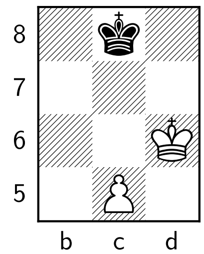


FIGURE 61. $1*$

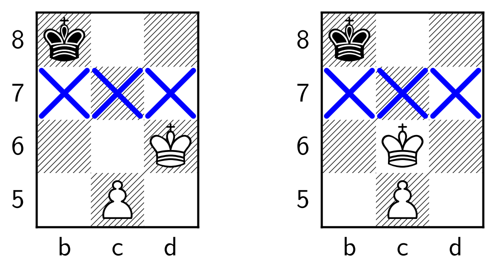


FIGURE 62. $\mathbf{On1}*$ and $\mathbf{On1}$

6. Concluding Remarks

There is some discrepancy between the way a chess player and a combinatorial game theorist understand a move as "good." In the following position, an experienced chess player would have no qualms playing either **. . . ♔c8** or **. . . ♔c7**, while the mathematician would not even consider **. . . ♔c8**, as it is dominated. Sure, some players would agree with the mathematicians, always playing **. . . ♔c7**, which is somehow intuitively stronger. But others might prefer the former, since the latter would probably just transpose. For a chess player, a win is a win and a draw is a draw. Both of these moves lead to simple draws, and therefore they are equivalent. When comparing two games, we (that is, chess players) might say that one is easier to win than another, but it does not make sense to say a game is "more winning" in the CGT sense. Still, the result that **blockade**$<$**FD** resonates with the chess player—the blockade is the more natural defensive position, and the mathematics demonstrates this. If nothing else, an expansion of this work will allow for an articulation of other similar ideas which may generally elude a definitive explanation. Indeed, after seeing the mathematics, it is

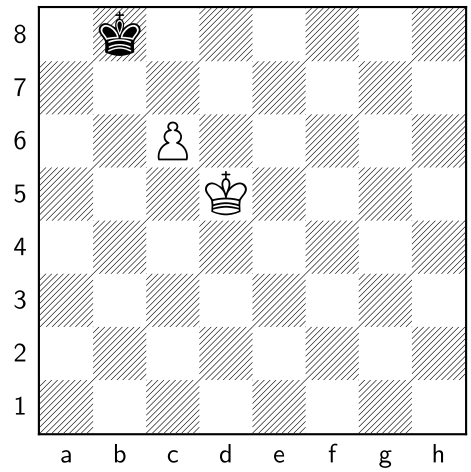


Figure 63. 2 Good Moves

not hard to clarify how these moves differ: if White had a tempo up his sleeve, **blockade** would offer Black the better chances.[17]

The previous section of this paper is quite mechanical. We begin by choosing a position to analyze and writing it in "bracket form." For each option, we ask, "Is this dominated? Reversible?" It is natural to wonder whether this process be automated. Answering these questions simply amounts to evaluating the outcome class of a specific game—a task which computers can certainly handle, at least for positions with only a few pieces. A simple algorithm could easily extend this work from the 3x4 board to the full 8x8 board. Unfortunately, things are not so simple moving further. The analysis here was simplified by two key factors. First, check was never a good move. This somewhat surprising fact is a feature which is unique to endgames with a King and a Pawn vs. a King. In order to expand this work further (say, King and 2 Pawns vs. King), a framework which allows entailing moves would be necessary. The second simplifying factor was the singular nature of the position. The focus was entirely on the promotion of the white Pawn. Unfortunately, as we add more pieces, we will not be able to analyze parts of the board as independent subgames. As Elkies points out, the chessboard is too small to allow for such a decomposition. Still, I believe the methods here can be applied to more complex positions. We will not necessarily find such elegant and familiar games, but we will find some truth nonetheless.

[17]Although it is not easy to give White a spare tempo without drastically altering the nature of the game.

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